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SPONTANEOUSLY GENERATED GLUONS AND GRAVITONS IN VECTOR AND TENSOR FIELD THEORIES

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Spontaneously Generated Gluons and Gravitons in Vector and Tensor Field Theories

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Abstract

We argue that Spontaneous Lorentz Invariance Violation (SLIV) could provide a dynamical approach to the Yang-Mills and tensor field gravity type theories with gluons and gravitons appearing as massless Goldstone and pseudo-Goldstone modes. The present thesis being related to this study is consisted of the several parts.

The first part includes the brief overview of the idea of SLIV which concerns a possible experimental motivation for Lorentz violation in general, from one hand, and theoretical argumentation providing the better understanding of the masslessness of gauge fields in Abelian quantum field theories, from the other.

The second chapter contains the analysis of non-Abelian case. The SLIV realized through a nonlinear vector field constraint of the type $Tr(A_\mu A^\mu) = \pm M^2$ (M is the proposed scale for Lorentz violation) is shown to generate massless vector Goldstone bosons, gauging the starting global internal symmetries in arbitrary relativistically invariant vector field theories. Actually, allowing the parameters in the Lagrangian to be adjusted so as to be consistent with this constraint, the

theory turns out to correspond to the gauged Yang-Mills theory, while the massless vector field appear as the Goldstone and pseudo-Goldstone vector bosons caused by SLIV. The theory looks essentially nonlinear and contains particular Lorentz (and CPT) violating couplings when expressed in terms of pure Goldstone vector modes. However, the model does not lead to physical Lorentz violation due to the simultaneously generated gauge invariance. The result are checked for some tree level processes.

The third chapter is basically related to the gravity case. In essence, the tensor field gravity theory, that mimics the linearized general relativity in Minkowski space-time, has been studied in which SLIV is realized through a nonlinear tensor field constraint $H_{\mu\nu}^2 = \pm M^2$. We show that such a SLIV pattern, due to which the true vacuum in the theory is chosen, induces massless tensor Goldstone modes some of which can naturally be associated with the physical graviton. Again as in the vector field case, this theory looks essentially nonlinear and contains a variety of Lorentz and CPT violating couplings. Nonetheless, all SLIV effects turn out to be strictly cancelled in all the lowest order processes considered, provided that the tensor field gravity theory is properly extended to general relativity (GR).

So, as we generally argue, the measurable effects of SLIV, induced by elementary vector or tensor fields, are related to the accompanying gauge symmetry breaking rather than to spontaneous Lorentz violation. The latter appears by itself to be physically unobservable, only resulting in a noncovariant gauge choice in an otherwise gauge invariant and Lorentz invariant theory. However, while Goldstonic vector and tensor field theories with exact local invariance are physically indistinguishable from conventional gauge theories, there might appear some principal distinctions if this local symmetry were slightly broken at very small distances in a way that could eventually allow one to differentiate between them observationally.

The fourth chapter summarizes the results established in the previous parts.

And finally thesis has a useful appendix including the details which are not presented in the main parts. These detail may help readers who want to carry out their own calculations in the Yang-Mills or Tensor Field Gravity type theories with the kinetic terms being different from the ordinary one.

Keywords: Origin of gauge symmetry; Spontaneous Lorentz invariance violation; Constrained gauge fields; Gauge fields and gravitons as Goldstone boson; Emergent gauge and gravity theories.

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List of Abbreviations

LIV (Lorentz Invariance Variolation)

SLIV (Spontaneous Lorentz Invariance Violation)

LI (Lorentz Invariance)

UHECRs (Ultra-Hight Energy Cosmic Rays)

CR (Cosmic Rays)

GR (General Relativity)

SR (Special Relativity)

SBLS (Spontaneous Breakdown of the Lorentz Symmetry)

VEV (Vacuum Expectation Value)

NG (Nambu-Goldstone)

Diff (Diffeomorphism)

PGB (Pseudo-Goldstone Bosons)

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Chapter 1

Brief Overview

Since 1905 the ideas of special relativity have been used not only as the fundamental theoretical principle, as it is taken presently in conventional Particle Physics and Quantum Field Theory, but almost in every aspect of our daily life. One could even state that the Lorentz Invariance is almost axiomatic postulate which the most of people believe in. Nevertheless, one could not declare so far, that Lorentz invariance is absolutely proven principle - instead, only following statement seems to be correct that according to experimental data we are tend to assume that Lorentz Invariance had been carrying with some known precision. In this chapter some motivation for the Spontaneous Lorentz Invariance Violation (SLIV) will be presented.

1.1 The motivation for SLIV

In the last decades there was an utterable interest in the breakdown of the Lorentz invariance, as a phenomenological possibility in the context of various quantum field theories as well as modified gravity and string theories[1].

It is an extremely challenging idea, that SLIV could provide a dynamical approach to quantum electrodynamics[2], gravity[3] and Yang-Mills[4] theories with photon, graviton and non-Abelian gauge fields appearing as massless Nambu-Goldstone (NG) bosons[5]. This idea has recently gained new impetus in the gravity sector - as for composite gravitons[6], in the case when gravitons identified with the NG modes of the symmetric two-index tensor field in the theory preserving a diffeomorphism (diff) invariance, apart from some non-invariant potential inducing spontaneous Lorentz violation[7].

Let us note at the same time that the experimental motivation for an explicit Lorentz violation is still very contradictory. For example one can find the publication[8] where the AGASA (Akeno Giant Air Shower Array) experimental group claims that for 14 years of operation they had observed 1000 ultra-high energy cosmic rays (UHECRs) above $10^{19}eV$ including eleven events above $10^{20}eV$ which is potentially in a conflict with the Greisen-Zatsepin-Kuzmin cut-off [9]. As could be expected, such an utterance produced a long cascade of a papers which was in an effort to produce a model without the cutoff. Nowadays it seems that the several events above GZK-cutoff that AGASA registered might be a consequence of

[1] Arkani-Hamed et al. (2004, 2007); Jacobson et al. (2006); Gripaios (2004); Jacobson et al. (2005); Kraus and Tomboulis (2002); Coleman and Glashow (1999)

[2] Bjorken (1963); Bialynicki-Birula (1963); Guralnik (1964)

[3] Atkatz (1978); Eguchi and Freund (1976); Ohanian (1969); Phillips (1966)

[4] Suzuki (1988); Terazawa et al. (1977)

[5] Nambu and Jona-Lasinio (1961)

[6] Berezhiani and Kancheli (2008)

[7] Kostelecky and Potting (2009); Carroll et al. (2009)

[8] Shinozaki and Teshima (2004); Shinozaki (2006)

[9] Greisen (1966); Zatsepin and Kuzmin (1966)

overestimating the primary energy in the surface array[10].

The work have been published recently claiming the violation of Lorentz invariance in high energy ions[11] however one could find the 'contra-article', suggesting a mistake based on miscalculations in Doppler effect[12].

Typically, analyzing any experiment with a given precision one always can take an ordinary theory and introduce some Lorentz violating terms in it. This has led to a great deal of theoretical speculation. For almost each Lorentz violating theoretical model one could tune the breakdown parameter to avoid the conflict with the experiment. So, it is extremely hard to choose the particular way to break Lorentz invariance in case of guidance only with experimental criteria.

However, luckily, we have some profound theoretical argumentation as well. As is well known, almost all the observed internal symmetries are more or less broken. Actually, our experience in particle physics gives us the knowledge that all the symmetries, apart from color and the electric charge, are broken in some extent. At the same time we know that at the early stage of particle physics the people tried to break the internal symmetry by introducing some additional breaking terms directly into the Lagrangian, just what some are often doing now with respect to Lorentz invariance. However, we have learned since then that the more acceptable (if not the only) way of the symmetry breaking is its spontaneous breakdown unless one does want to be under risk to completely lose control on the theory considered.

This is somehow surprising that Nature prefers to create the completely symmetric phase on a high energy scale and only afterwards spontaneously breaks it (on a relatively low one).

The ideas of spontaneously breakdown symmetry have been introduced in the condensed matter physics. It turns out that the idea is general and has fundamental meaning for the

[10] Capdevielle et al. (2009)

[11] Devasia (2010)

[12] Saathoff et al. (2011)

wide areas of applications culminating in the famous Goldstone theorem[13]. Despite the noticeable beauty and success of the this theorem the only known Goldstone boson that so far was found in particle physics is Nambu-Jona-Lasinio pion[14].

Nonetheless, it seems to be extremely interesting to consider all genuinely massless particles, like as photons or gravitons, as a Goldstonic particles that was argued a long ago (Bjorken, 2001). We briefly recall some of generic ingredients of this approach based on the four-fermion (current $\times$ current) interaction, where the Goldstonic gauge field may appear as a composite fermion-antifermion state. Unfortunately, owing to the lack of an initial gauge invariance in such models and the composite nature of the NG modes that appear, it is hard to explicitly demonstrate that these modes really form together a massless vector boson as a gauge field candidate. Actually, one must make a precise tuning of parameters in order to achieve the massless photon case. Rather, there are in general three separate massless NG modes, two of which may mimic the transverse photon polarizations, while the third one must be appropriately suppressed. In this connection, the more instructive laboratory for SLIV consideration proves to be some simple class of the QED type models[15] having from the outset a gauge invariant form whereas the spontaneous Lorentz violation is realized through the non-linear dynamical constraint that we are considering in the next section.

[13]Goldstone (1961); Goldstone et al. (1962)

[14]Nambu and Jona-Lasinio (1961)

[15]Nambu (1968); Chkareuli et al. (2004)

1.2 SLIV in QED

Now we have some rules of game or general principles. Actually in order to build the good theory in the symmetry-broken phase one has to use spontaneous Lorentz violation (rather than the direct one) and, besides, the initial gauge invariance is required. These instructions might seemed to be too general. However it is easy to build the theoretical constructions which would work indeed. All required ingredients are: the lagrangian of QED (just one neutral vector connected one fermion field by an ordinary coupling) and several additional polynomial terms of the vector field. Such a theory being first considered in the 60-ies in (Nambu, 1968) and in more detail relatively recently in (Chkareuli et al., 2004). The model is particularly interesting because of its ability to generate, on its own, the non-linear constraint for the vector field. The corresponding Lagrangian is

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu^2}{2}{A_\mu}^2 - \frac{\lambda^2}{4}\left({A_\mu}^2\right)^2 + \bar{\psi}(i\gamma\partial - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi \tag{1.1}$$

The free part of the Lagrangian is taken in the standard form such that Lorentz condition $\partial_\mu A^\mu(x) = 0$ automatically follows from the equation for the vector field A_μ when the self-interaction λA^4 term is absent in the Lagrangian. However, this term has to be added to implement the spontaneous breakdown of Lorentz symmetry $SO(1,3)$ down to the $SO(3)$ or $SO(1,2)$ for $\mu^2 > 0$ and $\mu^2 < 0$, respectively.

The equations of motion directly lead then to the constraint for the vector field (in case $\lambda \neq 0$):

$$A^2 = \frac{\mu^2}{\lambda}. \tag{1.2}$$

provided that the spin-1 or Lorentz condition is still required for vector field to be fulfilled

The similar models, while in other connection, have been also considered in the past[16]. By the using this nonlinear condition put on the vector field it is possible to express the entire

[16] Kostelecky and Lehnert (2001); Kostelecky and Samuel (1989); Righi and Venturi (1981, 1982); Nambu (1968),

Lagrangian in terms of Goldstonic modes that eventually leads the a construction, termed as a σ model for a QED in analogy with the non-linear σ model for pions. The model contains only two independent (and approximately transverse) vector Goldstone boson modes identified with the physical photon, and in the limit $M \to \infty$ is indistinguishable from conventional QED taken in the axial gauge.

Note that this correspondence with pion σ model may be somewhat suggestive, in view of the fact that pions are the only presently known Goldstones and their theory, chiral dynamics, is given by the non-linearly realized chiral $SU(2) \times SU(2)$ symmetry rather than by an ordinary linear model. The point is, however, that, in sharp contrast to the pion model, the non-linear QED theory, due to the starting gauge invariance involved, ensures that all the physical Lorentz violating effects are proved to be non-observable: the SLIV condition (1.2) is simply reduced to a possible gauge choice for the vector field A_{mu}, while the S-matrix remains unaltered under such a gauge convention. Really, this non-linear QED contains a plethora of particular Lorentz and CPT violating couplings when expressed in terms of the pure Goldstonic photon modes. However, contributions of these couplings to all physical processes involved are proved to be strictly cancelled, as was explicitly demonstrated in the tree approximation (Nambu, 1968). Some time ago, this result was extended to the one-loop approximation and for both time-like ($n^2 > 0$) and space-like ($n^2 < 0$) Lorentz violation Azatov and Chkareuli (2006). All the contributions to the photon-photon, photon-fermion and fermion-fermion interactions violating physical Lorentz invariance were shown to exactly cancel among themselves. This means that the constraint (1.2), having been treated as a non-linear gauge choice at the tree (classical) level, remains as a gauge condition when quantum effects are taken into account as well. So, in accordance with Nambus original conjecture, one can conclude that physical Lorentz invariance is left intact at least in the one-loop approximation, provided that we consider the standard gauge invariant QED Lagrangian (1.1) taken in flat Minkowski spacetime. Later this result was also confirmed for the spontaneously broken massive QED as well (Chkareuli and Kepuladze, 2007). Some interesting aspects of SLIV in nonlinear QED were considered in (Alfaro and Urrutia, 2010).

1.3 Specific aims of our study

Such a successful application of SLIV to Abelian theories with emergent massless photons appearing as vector Goldstone modes put the natural question concerning extension of the SLIV approach considered. Particularly:

Is it possible to expand our consideration to the non-Abelian case?

Should it provide us with some new physics behind?

If the Lorentz Invariance is spontaneously violated for the Yang-Mills types theories will there be an observable physical effects taking place?

And even more, is it possible to generalize our research and apply our SLIV ansatz to tensor field gravity type theories?

In the next chapters we will try to answer to the questions listed above, doing it step by step.

Chapter 2

Spontaneously Generated Gluons

The first models realizing the SLIV conjecture were based on the four fermion (current-current) interaction, where the gauge field appears as a fermion-antifermion pair composite state (Heisenberg, 1957), in complete analogy with the massless composite scalar field in the original Nambu-Jona-Lazinio model Nambu and Jona-Lasinio (1961). Unfortunately, owing to the lack of a starting gauge invariance in such models and the composite nature of the Goldstone modes which appear, it is hard to explicitly demonstrate that these modes really form together a massless vector boson as a gauge field candidate. Actually, one must make a precise tuning of parameters, including a cancelation between terms of different orders in the $1/N$ expansion (where N is the number of fermion species involved), in order to achieve the massless photon case (see, for example, the last paper in (Heisenberg, 1957)). Rather, there are in general three separate massless Goldstone modes, two of which may mimic the transverse photon polarizations, while the third one must be appropriately suppressed.

In this connection, a more instructive laboratory for SLIV consideration proves to be a simple class of QED type models[1] having from the outset a gauge invariant form. In these models the spontaneous Lorentz violation is realized through the nonlinear dynamical constraint $A_\mu A^\mu = n_\nu n^\nu M^2$ (where n_ν is a properly oriented unit Lorentz vector, $n_\nu n^\nu = \pm 1$, while M is the proposed SLIV scale) imposed on the starting vector field A_μ, in much the same way as it occurs for the corresponding scalar field in the nonlinear σ-model for pions Weinberg. Note that a correspondence with the nonlinear σ-model for pions may be somewhat suggestive, in view of the fact that pions are the only presently known Goldstones and their theory, chiral dynamics Weinberg, is given by the nonlinearly realized chiral $SU(2) \times SU(2)$ symmetry rather than by an ordinary linear σ-model. The above constraint means in essence that the vector field A_μ develops some constant background value $< A_\mu(x) > = n_\mu M$ and the Lorentz symmetry $SO(1,3)$ formally breaks down to $SO(3)$ or $SO(1,2)$ depending on the time-like ($n_\nu n^\nu > 0$) or space-like ($n_\nu n^\nu < 0$) nature of SLIV. This allows one to explicitly demonstrate that gauge theories, both Abelian and non-Abelian, can be interpreted as spontaneously broken theories, although the physical Lorentz invariance still remains intact.

However, the question naturally arises of whether a gauge symmetry is necessary to start with. If so, this would in some sense depreciate the latter approach as compared with those of the original composite models Heisenberg (1957), where a gauge symmetry was hoped to be derived (while this has not yet been achieved). Remarkably, as we will see, it happens that one does not need to specially postulate the starting gauge invariance, when considering the nonlinear σ-model type spontaneous Lorentz violation in the framework of an arbitrary relativistically invariant Lagrangian for elementary vector and matter fields, which are proposed only to possess some global internal symmetry. In the present article we start by a priori only assuming a global symmetry but no gauge invariance, taking all the terms

[1] **?**Nambu and Jona-Lasinio (1961); Katori et al. (2006); Bluhm and Kostelecky (2005); Azatov and Chkareuli (2006); Chkareuli and Kepuladze (2007); Chkareuli and Jejelava (2008)

in the Lagrangian allowed by Lorentz invariance. With such a Lagrangian, the vector field A_μ typically develops a non-zero vacuum expectation value,

$$< A_\mu(x) >= n_\mu M. \tag{2.1}$$

In the limit analogous to the approximation of the linear σ-model by the nonlinear σ-model, we get the nonlinear constraint:[2]

$$A^2 = n^2 M^2 \qquad (A^2 \equiv A_\mu A^\mu, \quad n^2 \equiv n_\nu n^\nu). \tag{2.2}$$

The chapter mostly rely on my publications (Chkareuli et al., 2008; Chkareuli and Jejelava, 2008). In this chapter is simply postulated that the existence of the constraint (2.2) is to be upheld by adjusting the parameters of the Lagrangian. It is shown that the SLIV conjecture, which is related to the condensation of a generic vector field or vector field multiplet, happens by itself to be powerful enough to impose gauge invariance, provided that I allow the corresponding Lagrangian density to be adjusted to ensure self-consistency without losing too many degrees of freedom. Due to the Lorentz violation, this theory acquires on its own a gauge-type invariance, which gauges the starting global symmetry of the interacting vector

[2] As it is mentioned already, some way to appreciate a possible origin for the supplementary condition (2.2) might be by the inclusion of a "standard" quartic vector field potential $U(A_\mu) = -\frac{m_A^2}{2}A^2 + \frac{\lambda_A}{4}(A^2)^2$ in the vector field Lagrangian, as can be motivated to some extent (Colladay and Kostelecky, 1998) from superstring theory. This potential inevitably causes the spontaneous violation of Lorentz symmetry in a conventional way, much as an internal symmetry violation is caused in a linear σ model for pions (Weinberg). As a result, one has a massive "Higgs" mode (with mass $\sqrt{2}m_A$) together with massless Goldstone modes associated with the photon. Furthermore, just as in the pion model, one can go from the linear model for the SLIV to the non-linear one by taking the limit $\lambda_A \to \infty$, $m_A^2 \to \infty$ (while keeping the ratio m_A^2/λ_A to be finite). This immediately leads to the constraint (2.2) for the vector potential A_μ with $n^2M^2 = m_A^2/\lambda_A$, as appears from the validity of its equation of motion. Another motivation for the nonlinear vector field constraint (2.2) might be an attempt to avoid an infinite self-energy for the electron in classical electrodynamics, as was originally suggested by Dirac (Dirac, 1951) and extended later to various vector field theory cases (Righi and Venturi, 1977).

and matter fields involved. In essence, the gauge invariance (with a proper gauge-fixing term) appears as a necessary condition for these vector fields not to be superfluously restricted in degrees of freedom. In fact the crucial equations (2.4) and (2.17) below express the relations needed to reduce the number of independent equations among the equations of motion and the constraint (2.2). But notice that it is not assumed gauge invariance to derive equations (2.4) and (2.17); The philosophy is to derive gauge invariance not to put it in. Due to the constraint (2.2), the true vacuum in a theory is chosen by the Lorentz violation, SLIV. The self-consistency problem to which we adjusted the couplings in the Lagrangian might have been avoided by using a Lagrange multiplier associated with the constraint (2.2). However it is rather the philosophy of the present article to look for consistency of the equations of motion and the constraint, without introducing such a Lagrange multiplier.

The next Sec. 2.1 consider the global Abelian symmetry case, which eventually appears as ordinary QED taken in a nonlinear gauge. While such a model for QED was considered before on its own[3], here is used the pure SLIV conjecture. Then in Sec. 2.2 the consideration is generalized to the global non-Abelian internal symmetry case and come to a conventional Yang-Mills theory with that symmetry automatically gauged. Specifically, we will see that in a theory with a symmetry group G having D generators not only the pure Lorentz symmetry $SO(1,3)$, but the larger accidental symmetry $SO(D,3D)$ of the Lorentz violating vector field constraint also happens to be spontaneously broken. As a result, although the pure Lorentz violation still generates only one true Goldstone vector boson, the accompanying pseudo-Goldstone vector bosons related to the $SO(D,3D)$ breaking also come into play properly completing the whole gauge field multiplet of the internal symmetry group taken. Remarkably, they appear to be strictly massless as well, being protected by the simultaneously generated non-Abelian gauge invariance. When expressed in terms of the pure Goldstone

[3](Nambu and Jona-Lasinio, 1961; Katori et al., 2006; Bluhm and Kostelecky, 2005; Azatov and Chkareuli, 2006; Chkareuli and Kepuladze, 2007; Chkareuli and Jejelava, 2008)

vector modes these theories, both Abelian and non-Abelian, look essentially nonlinear and contain Lorentz and CPT violating couplings. However, due to cancellations, they appear to be physically indistinguishable from the conventional QED and Yang-Mills theories. On the other hand, their generic, SLIV induced, gauge invariance could of course be broken by some high-order operators, stemming from very short gravity-influenced distances that would lead to the physical Lorentz violation. This and some other of the conclusions are discussed in the final part.

2.1 Abelian theory

Suppose first that there is only one vector field A_μ and one complex matter field ψ, a charged fermion or scalar, in a theory given by a general Lorentz invariant Lagrangian $L(A, \psi)$ with the corresponding global $U(1)$ charge symmetry imposed. Before proceeding further, note first that, while a conventional variation principle requires the equation of motion

$$\frac{\partial L}{\partial A_\mu} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu A_\mu)} = 0 \tag{2.3}$$

to be satisfied, the vector field A_μ, both massive and massless, still contains one superfluous component which is usually eliminated by imposing some supplementary condition. This is typically imposed by taking the 4-divergence of the Euler equation (2.3). Such a condition for the massive QED case (with the gauge invariant $F_{\mu\nu}F^{\mu\nu}$ form for the vector field kinetic term) is known to be the spin-1 or Lorentz condition $\partial_\mu A^\mu = 0$, while for the conventional massless QED many other conditions (gauges) may alternatively be taken.

Let us now subject the vector field $A_\mu(x)$ in a general Lagrangian $L(A_\mu, \psi)$ to the SLIV constraint (2.2), which presumably chooses the true vacuum in a theory. Once the SLIV constraint is imposed, any extra supplementary condition is no longer possible, since this would superfluously restrict the number of degrees of freedom for the vector field which is inadmissible. In fact a further reduction in the number of independent A_μ components would make it impossible to set the required initial conditions in the appropriate Cauchy problem and, in quantum theory, to choose self-consistent equal-time commutation relations[4] (Ogievetsky and Polubarinov, 1965). It is also well-known Weinberg that there is no way to construct a massless field A_μ, which transforms properly as a 4-vector, as a linear combination

[4]For example the need for more than two degrees of freedom is well-known for a massive vector field and for quantum electrodynamics. In the massive vector field case there are three physical spin-1 states to be described by the A_μ, whereas for QED, apart from the two physical (transverse) photon spin states, one formally needs one more component in the A_μ (A_0 or A_3) as the Lagrange multiplier to get the Gauss law. So, in both cases only one component in the A_μ may be eliminated.

of creation and annihilation operators for helicity ± 1 states.

Under this assumption of not getting too many constraints[5], it is possible to derive gauge invariance. Since the 4-divergence of the vector field Euler equation (2.3) should be zero if the equations of motion are used, it means that this divergence must be expressible as a sum over the equations of motion multiplied by appropriate quantities. This implies that, without using the equations of motion but still using the constraint (2.2), we have an identity for the vector and matter (fermion field, for definiteness) fields of the following type:

$$\begin{aligned} \partial_\mu \left(\frac{\partial L}{\partial A_\mu} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu A_\mu)} \right) \equiv & \left(\frac{\partial L}{\partial A_\mu} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu A_\mu)} \right)(c)A_\mu + \\ & + \left(\frac{\partial L}{\partial \psi} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu \psi)} \right)(it)\psi + \\ & + \overline{\psi}(-it) \left(\frac{\partial L}{\partial \overline{\psi}} - \partial_\nu \frac{\partial L}{\partial(\partial_\nu \overline{\psi})} \right). \end{aligned} \tag{2.4}$$

Here the coefficients c and t of the Eulerians on the right-hand side (which vanish by themselves when the equations of motion are fulfilled) are some dimensionless constants whose particular values are conditioned by the starting Lagrangian $L(A_\mu, \psi)$ taken, for simplicity, with renormalisable coupling constants. This identity (2.4) implies the invariance of L under the vector and fermion field local transformations whose infinitesimal form is given by[6]

$$\delta A_\mu = \partial_\mu \omega + c\omega A_\mu, \quad \delta\psi = it\omega\psi \tag{2.5}$$

where $\omega(x)$ is an arbitrary function, only being restricted by the requirement to conform with the nonlinear constraint (2.2). Conversely, the identity (2.4) in its turn follows from the invariance of the Lagrangian L under the transformations (2.5). Both direct and con-

[5]The fact that there is a threat of too many supplementary conditions (an inconsistency) is because we have chosen not to put a Lagrange multiplier term for the constraint (2.2) into Eq. (2.3). Had we explicitly introduced such a Lagrange multiplier term, $F(x)(A^2 - n^2 M^2)$, into the Lagrangian L, the equation of motion for the vector field A_μ would have changed, so that the 4-divergence of this equation would now determine the Lagrange multiplier function $F(x)$ rather than satisfy the identity (2.4) appearing below.

[6]Actually, one can confirm this proposition by expanding the action with the transformed Lagrangian density $\int d^4x L(A', \psi')$ in terms of functional derivatives and then using the identity equation (2.4).

verse assertions are in fact particular cases of Noether's second theorem (Noether, 1918). Apart from this invariance, one has now to confirm that the transformations (2.5) in fact form an Abelian symmetry group. Constructing the corresponding Lie bracket operation $(\delta_1\delta_2 - \delta_2\delta_1)$ for two successive vector field variations one could find that, while the fermion transformation in (2.5) is an ordinary Abelian local one with zero Lie bracket, for the vector field transformations there appears a non-zero result

$$(\delta_1\delta_2 - \delta_2\delta_1)A_\mu = c(\omega_1\partial_\mu\omega_2 - \omega_2\partial_\mu\omega_1) \tag{2.6}$$

unless the coefficient $c = 0$. Note also that for non-zero c the variation of A_μ given by (2.6) is an essentially arbitrary vector function. Such a freely varying A_μ is only consistent with a trivial Lagrangian (i.e. $L = const$). Thus, in order to have a non-trivial Lagrangian, it is necessary to have $c = 0$ and the theory then possesses an Abelian local symmetry[7].

Thus one could figure out how the choice of a true vacuum conditioned by the SLIV constraint (2.2) enforces the modification of the Lagrangian L, so as to convert the starting global $U(1)$ charge symmetry into a local one (2.5). Otherwise, the theory would superfluously restrict the number of degrees of freedom for the vector field and that would be inadmissible. This SLIV induced local Abelian symmetry (2.5) now allows the Lagrangian L to be determined in full. For a minimal theory with renormalisable coupling constants, it is in fact the conventional QED Lagrangian which we eventually come to:

$$L(A_\mu, \psi) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma\partial - m)\psi - eA_\mu\overline{\psi}\gamma^\mu\psi \tag{2.7}$$

with the SLIV constraint $A^2 = n^2M^2$ imposed on the vector field A_μ. In the derivation made, we were only allowed to use gauge transformations consistent with the constraint (2.2) which now plays the role of a gauge-fixing term for the resulting gauge invariant theory[8]

[7] In Sec. 2.2 will be shown that non-zero c-type coefficients appear in the non-Abelian internal symmetry case, resulting eventually in a Yang-Mills gauge invariant theory.

[8] As indicated in refs. ?Dirac (1951), the SLIV constraint equation for the corresponding finite gauge

(2.7). Note that a quartic potential $U(A_\mu)$ of the type discussed in footnote 1 would give vanishing contributions on both sides of Eq. (2.4), when the nonlinear constraint (2.2) with the SLIV scale M^2 given in the footnote is imposed. Furthermore the contribution of such a potential to the Lagrangian (2.7) would then reduce to an inessential constant.

One can rewrite the Lagrangian $L(A_\mu, \psi)$ in terms of the physical photons now identified as being the SLIV generated vector Goldstone bosons. For this purpose let us take the following handy parameterization for the vector potential A_μ in the Lagrangian L:

$$A_\mu = a_\mu + \frac{n_\mu}{n^2}(n \cdot A) \qquad (n \cdot A \equiv n_\nu A^\nu) \tag{2.8}$$

where a_μ is the pure Goldstonic mode satisfying

$$n \cdot a = 0, \qquad (n \cdot a \equiv n_\nu a^\nu) \tag{2.9}$$

while the effective "Higgs" mode (or the A_μ component in the vacuum direction) is given by the scalar product $n \cdot A$. Substituting this parameterization (2.8) into the vector field constraint (2.2), one comes to the equation for $n \cdot A$:

$$n \cdot A \ = (M^2 - n^2 a^2)^{\frac{1}{2}} = M - \frac{n^2 a^2}{2M} + O(1/M^2) \tag{2.10}$$

where $a^2 = a_\mu a^\mu$ and taking, for definiteness, the positive sign for the square root and expanding it in powers of a^2/M^2. Putting then the parametrization (2.8) with the SLIV constraint (2.10) into our basic gauge invariant Lagrangian (2.7), one comes to the truly Goldstonic model for QED. This model might seem unacceptable since it contains, among other terms, the inappropriately large Lorentz violating fermion bilinear $eM\overline{\psi}(\gamma \cdot n/n^2)\psi$, which appears when the expansion (2.10) is applied to the fermion current interaction term in the Lagrangian L (2.7). However, due to local invariance of the Lagrangian (2.7), this term

function $\omega(x)$, $(A_\mu + \partial_\mu \omega)(A^\mu + \partial^\mu \omega) = n^2 M^2$, appears to be mathematically equivalent to the classical Hamilton-Jacobi equation of motion for a charged particle. Thus, this equation should have a solution for some class of gauge functions $\omega(x)$, inasmuch as there is a solution to the classical problem.

can be gauged away by making an appropriate redefinition of the fermion field according to

$$\psi \rightarrow e^{ieM(x\cdot n/n^2)}\psi \tag{2.11}$$

through which the $eM\overline{\psi}(\gamma\cdot n/n^2)\psi$ term is exactly canceled by an analogous term stemming from the fermion kinetic term. So, one eventually arrives at the essentially nonlinear SLIV Lagrangian for the Goldstonic a_μ field of the type (taken to first order in a^2/M^2)

$$\begin{aligned} L(a_\mu,\psi) \;=\; & -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{2}\delta(n\cdot a)^2 - \frac{1}{4}f_{\mu\nu}h^{\mu\nu}\frac{n^2a^2}{M} + \\ & +\overline{\psi}(i\gamma\partial + m)\psi - ea_\mu\overline{\psi}\gamma^\mu\psi + \frac{en^2a^2}{2M}\overline{\psi}(\gamma\cdot n)\psi. \end{aligned} \tag{2.12}$$

We have denoted its field strength tensor by $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, while $h_{\mu\nu} = n^\mu\partial^\nu - n^\nu\partial^\mu$ is a new SLIV oriented differential tensor acting on the infinite series in a^2 coming from the expansion of the effective "Higgs" mode (3.14), from which we have only included the first order term $-n^2a^2/2M$ throughout the Lagrangian $L(a_\mu,\psi)$. One has also explicitly introduced the orthogonality condition $n\cdot a = 0$ into the Lagrangian through the second term, which can be treated as the gauge fixing term (taking the limit $\delta\rightarrow\infty$). Furthermore we have retained the notation ψ for the redefined fermion field.

This nonlinear QED model was first studied on its own by Nambu long ago (**?**). As one can see, the model contains the massless vector Goldstone boson modes (keeping the massive "Higgs" mode frozen), and in the limit $M\rightarrow\infty$ is indistinguishable from conventional QED taken in the general axial (temporal or pure axial) gauge. So, for this part of the Lagrangian $L(a_\mu,\psi)$ given by the zero-order terms in $1/M$, the spontaneous Lorentz violation simply corresponds to a non-covariant gauge choice in an otherwise gauge invariant (and Lorentz invariant) theory. Remarkably, also all the other (first and higher order in $1/M$) terms in $L(a_\mu,\psi)$ (2.12), though being by themselves Lorentz and CPT violating ones, appear not to cause physical SLIV effects due to strict cancellations in the physical processes involved. So, the non-linear constraint (2.2) applied to the standard QED Lagrangian (2.7) appears in fact to be a possible gauge choice, while the S-matrix remains unaltered under such a gauge convention. This conclusion was first confirmed at the tree level (?) and recently extended to the one-loop approximation (Azatov and Chkareuli, 2006). All the one-loop contributions to the photon-photon, photon-fermion and fermion-fermion interactions violating Lorentz invariance were shown to be exactly canceled with each other, in the manner observed earlier for the simplest tree-order diagrams. This suggests that the vector field constraint $A^2 = n^2M^2$, having been treated as a nonlinear gauge choice at the tree (classical) level, remains as just a gauge condition when quantum effects are taken into account as well.

To resume let's recall the steps made in the derivation above. We started with the most general Lorentz invariant Lagrangian $L(A_\mu, \psi)$, proposing only a global internal $U(1)$ symmetry for the charged matter fields involved. The requirement for the vector field equations of motion to be compatible with the true vacuum chosen by the SLIV (2.2) led us to the necessity for the identity (2.4) to be satisfied by the Lagrangian L. According to Noether's second theorem Noether (1918), this identity implies the invariance of the Lagrangian L under the $U(1)$ charge gauge transformations of all the interacting fields. And, finally, this local symmetry allows us to completely establish the underlying theory, which appears to be standard QED (2.7) taken in the nonlinear gauge (2.2) or the nonlinear σ model-type QED in a general axial gauge - both preserving physical Lorentz invariance.

2.2 Non-Abelian theory

Now the discussion is extended to the non-Abelian global internal symmetry case for a general Lorentz invariant Lagrangian $\mathcal{L}(\boldsymbol{A}_\mu, \boldsymbol{\psi})$ for the vector and matter fields involved. This symmetry is given by a general group G with D generators t^α

$$[t_\alpha, t_\beta] = ic_{\alpha\beta\gamma}t_\gamma, \quad Tr(t_\alpha t_\beta) = \delta_{\alpha\beta} \quad (\alpha, \beta, \gamma = 0, 1, ..., D-1) \tag{2.13}$$

where $c_{\alpha\beta\gamma}$ are the structure constants of G. The corresponding vector fields, which transform according to the adjoint representation of G, are given in the matrix form $\boldsymbol{A}_\mu = \boldsymbol{A}_\mu^\alpha t_\alpha$. The matter fields (fermions or scalars) are, for definiteness, taken in the fundamental representation column $\boldsymbol{\psi}^\sigma$ $(\sigma = 0, 1, ..., d-1)$ of G. Let us again, as in the above Abelian case, subject the vector field multiplet $\boldsymbol{A}_\mu^\alpha(x)$ to a SLIV constraint of the form

$$Tr(\boldsymbol{A}_\mu \boldsymbol{A}^\mu) = \boldsymbol{n}^2 M^2, \quad \boldsymbol{n}^2 \equiv \boldsymbol{n}_\mu^\alpha \boldsymbol{n}^{\mu,\alpha} = \pm 1, \tag{2.14}$$

that presumably chooses the true vacuum in a theory. Here, as usual, we sum over repeated indices. This covariant constraint is not only the simplest one, but the only possible SLIV condition which could be written for the vector field multiplet $\boldsymbol{A}_\mu^\alpha$ and not be superfluously restricted (see discussion below).

Although here is only proposed the $SO(1,3) \times G$ invariance of the Lagrangian $\mathcal{L}(\boldsymbol{A}_\mu, \boldsymbol{\psi})$, the chosen SLIV constraint (2.14) in fact possesses a much higher accidental symmetry $SO(D, 3D)$ determined by the dimensionality D of the G adjoint representation to which the vector fields $\boldsymbol{A}_\mu^\alpha$ belong[9]. This symmetry is indeed spontaneously broken at a scale M

$$< \boldsymbol{A}_\mu^\alpha(x) > \ = \boldsymbol{n}_\mu^\alpha M \tag{2.15}$$

with the vacuum direction given now by the 'unit' rectangular matrix $\boldsymbol{n}_\mu^\alpha$ describing simultaneously both of the generalized SLIV cases, time-like ($SO(D,3D) \to SO(D-1,3D)$) or space-like ($SO(D,3D) \to SO(D,3D-1)$) respectively, depending on the sign of $\boldsymbol{n}^2 \equiv \boldsymbol{n}_\mu^\alpha \boldsymbol{n}^{\mu,\alpha} = \pm 1$. This matrix has in fact only one non-zero element for both cases, subject to the appropriate $SO(D,3D)$ rotation. They are, specifically, $\boldsymbol{n}_0^0$ or $\boldsymbol{n}_3^0$ provided that the vacuum expectation value (2.15) is developed along the $\alpha = 0$ direction in the internal space and along the $\mu = 0$ or $\mu = 3$ direction respectively in the ordinary four-dimensional one. As we shall soon see, in response to each of these two breakings, side by side with one true vector Goldstone boson corresponding to the spontaneous violation of the actual $SO(1,3) \otimes G$ symmetry of the Lagrangian $\mathcal{L}$, $D-1$ vector pseudo-Goldstone bosons (PGB) related to a breaking of the accidental $SO(D,3D)$ symmetry of the constraint (2.14) per se are also produced[10]. Remarkably, in contrast to the familiar scalar PGB case Weinberg, the vector PGBs remain strictly massless being protected by the simultaneously generated non-Abelian gauge invariance. Together with the above true vector Goldstone boson, they just complete the whole gauge field

[9]Actually, in the same way as in the Abelian case[1], such a SLIV constraint (2.14) might be related to the minimisation of some $SO(D,3D)$ invariant vector field potential $\mathcal{U}(\boldsymbol{A}_\mu) = -\frac{m_A^2}{2} Tr(\boldsymbol{A}_\mu \boldsymbol{A}^\mu) + \frac{\lambda_A}{4}[Tr(\boldsymbol{A}_\mu \boldsymbol{A}^\mu)]^2$ followed by taking the limit $m_A^2 \to \infty$, $\lambda_A \to \infty$ (while keeping the ratio m_A^2/λ_A finite). Notably, the inclusion into this potential of another possible, while less symmetrical, four-linear self-interaction term of the type $(\lambda_A'/4) Tr(\boldsymbol{A}_\mu \boldsymbol{A}^\mu \boldsymbol{A}_\nu \boldsymbol{A}^\nu)$ would lead, as one can easily confirm, to an unacceptably large number ($4D$) of vector field constraints at the potential minimum.

[10]Note that in total there appear $4D-1$ pseudo-Goldstone modes, complying with the number of broken generators of $SO(D,3D)$, both for time-like and space-like SLIV. From these $4D-1$ pseudo-Goldstone modes, $3D$ modes correspond to the D three component vector states as will be shown below, while the remaining $D-1$ modes are scalar states which will be excluded from the theory. In fact $D-r$ actual scalar Goldstone bosons (where r is the rank of the group G), arising from the spontaneous violation of G, are contained among these excluded scalar states.

multiplet of the internal symmetry group G.

Let us now turn to the possible supplementary conditions which can be imposed on the vector fields in a general Lagrangian $\mathcal{L}(\boldsymbol{A}_\mu, \boldsymbol{\psi})$, in order to finally establish its form. While generally D supplementary conditions may be imposed on the vector field multiplet $\boldsymbol{A}_\mu^\alpha$, one of them in the case considered is in fact the SLIV constraint (2.14). One might think that the other conditions would appear by taking 4-divergences of the equations of motion

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{A}_\mu^\alpha} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \boldsymbol{A}_\mu^\alpha)} = 0, \tag{2.16}$$

which are determined by a variation of the Lagrangian $\mathcal{L}$. The point is, however, that due to the G symmetry this operation would lead, on equal terms, to D independent conditions thus giving in total, together with the basic SLIV constraint (2.14), $D+1$ constraints for the vector field multiplet $\boldsymbol{A}_\mu^\alpha$ which is inadmissible. Therefore, as in the above Abelian case, the 4-divergences of the Euler equations (2.16) should not produce supplementary conditions at all once the SLIV occurs. This means again that such 4-divergences should be arranged to vanish (though still keeping the global G symmetry) either identically or as a result of the equations of motion for vector and matter fields (fermion fields for definiteness) thus implying that, in the absence of these equations, there must hold a general identity of the type

$$\begin{aligned}
\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{A}_\mu^\alpha} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \boldsymbol{A}_\mu^\alpha)} \right) \equiv & \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{A}_\mu^\beta} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \boldsymbol{A}_\mu^\beta)} \right) C_{\alpha\beta\gamma} \boldsymbol{A}_\mu^\gamma + \\
& + \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\psi}} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \boldsymbol{\psi})} \right) (iT_\alpha) \boldsymbol{\psi} + \\
& + \overline{\boldsymbol{\psi}}(-iT_\alpha) \left(\frac{\partial \mathcal{L}}{\partial \overline{\boldsymbol{\psi}}} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \overline{\boldsymbol{\psi}})} \right).
\end{aligned} \tag{2.17}$$

The coefficients $C_{\alpha\beta\gamma}$ and T_α of the Eulerians on the right-hand side of the identity (2.17) can readily be identified with the structure constants $c_{\alpha\beta\gamma}$ and generators t_α (2.13) of the group G. This follows because the right hand side of the identity (2.17) must transform in the same way as the left hand side, which transforms as the adjoint representation of G. Note that these coefficients consist of dimensionless constants corresponding to the starting 'minimal' Lagrangian $\mathcal{L}(\boldsymbol{A}_\mu, \boldsymbol{\psi})$ which is taken, for simplicity, with renormalisable coupling constants. According to Noether's second theorem Noether (1918), the identity (2.17) again means the invariance of $\mathcal{L}$ under the vector and fermion field local transformations having the infinitesimal form

$$\delta \boldsymbol{A}_\mu^\alpha = \partial_\mu \omega^\alpha + C_{\alpha\beta\gamma}\omega^\beta \boldsymbol{A}_\mu^\gamma, \quad \delta\boldsymbol{\psi} \ = iT_\alpha \omega^\alpha \boldsymbol{\psi} \tag{2.18}$$

where $\omega^\alpha(x)$ are arbitrary functions only being restricted, again as in the above Abelian case, by the requirement to conform with the corresponding nonlinear constraint (2.14).

Note that the existence of the starting global G symmetry in the theory is important for our consideration, since without such a symmetry the basic identity (2.17) would be written with arbitrary coefficients $C_{\alpha\beta\gamma}$ and T_α. Then this basic identity may be required for only some particular vector field $\boldsymbol{A}_\mu^{\alpha_0}$ rather than for the entire set $\boldsymbol{A}_\mu^\alpha$. This would eventually lead to the previous pure Abelian theory case just for this $\boldsymbol{A}_\mu^{\alpha_0}$ component leaving aside all the other ones. Just the existence of the starting global symmetry G ensures a non-Abelian group-theoretical solution for the local transformations (2.18) in the theory.

So, we have shown that in the non-Abelian internal symmetry case, as well as in the Abelian case, the imposition of the SLIV constraint (2.14) converts the starting global symmetry G into the local one G_{loc}. Otherwise, the theory would superfluously restrict the number of degrees of freedom for the vector field multiplet $\boldsymbol{A}_\mu^\alpha$, which would certainly not be allowed. This SLIV induced local non-Abelian symmetry (2.18) now completely determines the Lagrangian $\mathcal{L}$, following the standard procedure (see, for example, (Mohapatra)). For a minimal theory with renormalisable coupling constants, this corresponds in fact to a conventional Yang-Mills type Lagrangian

$$\mathcal{L}(\boldsymbol{A}_\mu, \psi) = -\frac{1}{4}Tr(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}) + \overline{\boldsymbol{\psi}}(i\gamma\partial - m)\boldsymbol{\psi} + g\overline{\boldsymbol{\psi}}\boldsymbol{A}_\mu\gamma^\mu\boldsymbol{\psi} \tag{2.19}$$

(where $\boldsymbol{F}_{\mu\nu} \ = \ \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu - ig[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]$ and g stands for the universal coupling constant in the theory) with the SLIV constraint (2.14) imposed. These constrained gauge fields $\boldsymbol{A}_\mu^\alpha$

contain, as we directly confirm below, one true Goldstone and $D-1$ pseudo-Goldstone vector bosons, corresponding to the spontaneous violation of the accidental $SO(D,3D)$ symmetry of the constraint (2.14).

Actually, as in the above Abelian case, after the explicit use of the corresponding SLIV constraint (2.14), which is so far the only supplementary condition for the vector field multiplet $\boldsymbol{A}_\mu^\alpha$, one can identify the pure Goldstone field modes $\boldsymbol{a}_\mu^\alpha$ as follows:

$$\boldsymbol{A}_\mu^\alpha = \boldsymbol{a}_\mu^\alpha + \frac{\boldsymbol{n}_\mu^\alpha}{\boldsymbol{n}^2}(\boldsymbol{n}\cdot\boldsymbol{A}), \quad \boldsymbol{n}\cdot\boldsymbol{a} \equiv \boldsymbol{n}_\mu^\alpha \boldsymbol{a}^{\mu,\alpha} \ = 0. \tag{2.20}$$

At the same time an effective "Higgs" mode (i.e., the $\boldsymbol{A}_\mu^\alpha$ component in the vacuum direction $\boldsymbol{n}_\mu^\alpha$) is given by the product $\boldsymbol{n}\cdot\boldsymbol{A} \equiv \boldsymbol{n}_\mu^\alpha \boldsymbol{A}^{\mu,\alpha}$ determined by the SLIV constraint

$$\boldsymbol{n}\cdot\boldsymbol{A} \ = \left[M^2 - \boldsymbol{n}^2\boldsymbol{a}^2\right]^{\frac{1}{2}} = M - \frac{\boldsymbol{n}^2\boldsymbol{a}^2}{2M} + O(1/M^2). \tag{2.21}$$

where $\boldsymbol{a}^2 \ = \boldsymbol{a}_\nu^\alpha \boldsymbol{a}^{\nu,\alpha}$. As earlier in the Abelian case, we take the positive sign for the square root and expand it in powers of $\boldsymbol{a}^2/M^2$. Note that, apart from the pure vector fields, the general Goldstonic modes $\boldsymbol{a}_\mu^\alpha$ contain $D-1$ scalar fields, $\boldsymbol{a}_0^{\alpha'}$ or $\boldsymbol{a}_3^{\alpha'}$ ($\alpha' = 1...D-1$), for the time-like ($\boldsymbol{n}_\mu^\alpha = n_0^0 g_{\mu 0}\delta^{\alpha 0}$) or space-like ($\boldsymbol{n}_\mu^\alpha = n_3^0 g_{\mu 3}\delta^{\alpha 0}$) SLIV respectively. They can be eliminated from the theory if one imposes appropriate supplementary conditions on the $\boldsymbol{a}_\mu^\alpha$ fields which are still free of constraints. Using their overall orthogonality (2.20) to the physical vacuum direction $\boldsymbol{n}_\mu^\alpha$, one can formulate these supplementary conditions in terms of a general axial gauge for the entire $\boldsymbol{a}_\mu^\alpha$ multiplet

$$n\cdot\boldsymbol{a}^\alpha \equiv n_\mu \boldsymbol{a}^{\mu,\alpha} = 0, \quad \alpha = 0,1,...D-1. \tag{2.22}$$

Here n_μ is the unit Lorentz vector, analogous to that introduced in the Abelian case, which is now oriented in Minkowskian space-time so as to be parallel to the vacuum matrix[11] $\boldsymbol{n}_\mu^\alpha$. As a result, apart from the "Higgs" mode excluded earlier by the above orthogonality condition

[11]For such a choice the simple identity $\boldsymbol{n}_\mu^\alpha \equiv \frac{n\cdot\boldsymbol{n}^\alpha}{n^2} n_\mu$ holds, showing that the rectangular vacuum matrix $\boldsymbol{n}_\mu^\alpha$ has the factorized "two-vector" form.

(2.20), all the other scalar fields are also eliminated, and only the pure vector fields, $\boldsymbol{a}_i^\alpha$ ($i = 1, 2, 3$) or $\boldsymbol{a}_{\mu'}^\alpha$ ($\mu' = 0, 1, 2$) for time-like or space-like SLIV respectively, are left in the theory. Clearly, the components $\boldsymbol{a}_i^{\alpha=0}$ and $\boldsymbol{a}_{\mu'}^{\alpha=0}$ correspond to the Goldstone boson, for each type of SLIV respectively, while all the others (for $\alpha = 1...D-1$) are vector PGBs.

We now show that these Goldstonic vector fields, denoted generally as $\boldsymbol{a}_\mu^\alpha$ but with the supplementary conditions (2.22) understood, appear truly massless in the SLIV inspired gauge invariant Lagrangian $\mathcal{L}$ (2.19) subject to the SLIV constraint (2.14). Actually, substituting the parameterization (2.20) with the SLIV constraint (2.21) into the Lagrangian (2.19), one is led to a highly nonlinear Yang-Mills theory in terms of the pure Goldstonic modes $\boldsymbol{a}_\mu^\alpha$. However, as in the above Abelian case, one should first use the local invariance of the Lagrangian $\mathcal{L}$ to gauge away the apparently large Lorentz violating terms, which appear in the theory in the form of fermion and vector field bilinears. As one can readily see, they stem from the expansion (2.21) when it is applied to the couplings $g\overline{\boldsymbol{\psi}}\boldsymbol{A}_\mu\gamma^\mu\boldsymbol{\psi}$ and $-\frac{1}{4}g^2Tr([\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]^2)$ respectively in the Lagrangian (2.19). Analogously to the Abelian case, we make the appropriate redefinitions of the fermion ($\boldsymbol{\psi}$) and vector ($\boldsymbol{a}_\mu \equiv \boldsymbol{a}_\mu^\alpha t^\alpha$) field multiplets:

$$\boldsymbol{\psi} \to U(\omega)\boldsymbol{\psi} \ , \quad \boldsymbol{a}_\mu \to U(\omega)\boldsymbol{a}_\mu U(\omega)^\dagger, \quad U(\omega) = e^{igM(x\cdot\boldsymbol{n}^\alpha/\boldsymbol{n}^2)\boldsymbol{t}^\alpha}. \tag{2.23}$$

Since the phase of the transformation matrix $U(\omega)$ is linear in the space-time coordinate, the following equalities are evidently satisfied:

$$\partial_\mu U(\omega) = igM\boldsymbol{n}_\mu U(\omega) = igMU(\omega)\boldsymbol{n}_{\boldsymbol{\mu}}, \quad \boldsymbol{n}_{\boldsymbol{\mu}} \equiv \boldsymbol{n}_\mu^\alpha t^\alpha. \tag{2.24}$$

One can readily confirm that the above-mentioned Lorentz violating terms are thereby cancelled with the analogous bilinears stemming from their kinetic terms. So, the final Lagrangian for the Goldstonic Yang-Mills theory takes the form (to first order in $(\boldsymbol{a}^2/M^2)$

$$\begin{aligned}\mathcal{L}(\boldsymbol{a}_\mu^\alpha, \boldsymbol{\psi}) &= -\frac{1}{4}Tr(\boldsymbol{f}_{\mu\nu}\boldsymbol{f}^{\mu\nu}) - \frac{1}{2}\boldsymbol{\delta}(n\cdot\boldsymbol{a}^\alpha)^2 + \frac{1}{4}Tr(\boldsymbol{f}_{\mu\nu}\boldsymbol{h}^{\mu\nu})\frac{\boldsymbol{n}^2\boldsymbol{a}^2}{M} + \\ &+\overline{\boldsymbol{\psi}}(i\gamma\partial - m)\boldsymbol{\psi} + g\overline{\boldsymbol{\psi}}\boldsymbol{a}_\mu\gamma^\mu\boldsymbol{\psi} - \frac{g\boldsymbol{n}^2\boldsymbol{a}^2}{2M}\overline{\boldsymbol{\psi}}(\gamma\cdot\boldsymbol{n})\boldsymbol{\psi}. \qquad (2.25)\end{aligned}$$

Here the tensor $\boldsymbol{f}_{\mu\nu}$ is, as usual, $\boldsymbol{f}_{\mu\nu} = \partial_\mu\boldsymbol{a}_\nu - \partial_\nu\boldsymbol{a}_\mu - ig[\boldsymbol{a}_\mu, \boldsymbol{a}_\nu]$, while $\boldsymbol{h}_{\mu\nu}$ is a new SLIV oriented tensor of the type

$$\boldsymbol{h}_{\mu\nu} = \boldsymbol{n}_\mu\partial_\nu - \boldsymbol{n}_\nu\partial_\mu + ig([\boldsymbol{n}_\mu, \boldsymbol{a}_\nu] - [\boldsymbol{n}_\nu, \boldsymbol{a}_\mu])$$

acting on the infinite series in $\boldsymbol{a}^2$ coming from the expansion of the effective “Higgs” mode (2.21), from which we have only included the first order term $-\boldsymbol{n}^2\boldsymbol{a}^2/2M$ throughout the Lagrangian $\mathcal{L}(\boldsymbol{a}_\mu^\alpha, \boldsymbol{\psi})$. We have explicitly introduced the (axial) gauge fixing term into the Lagrangian, corresponding to the supplementary conditions (2.22) imposed. We have also retained the original notations for the fermion and vector fields after the transformations (2.23).

The theory we here derived is in essence a generalization of the nonlinear QED model (**?**) for the non-Abelian case. As one can see, this theory contains the massless vector Goldstone and pseudo-Goldstone boson multiplet $\boldsymbol{a}_\mu^\alpha$ gauging the starting global symmetry G and, in the limit $M \to \infty$, is indistinguishable from conventional Yang-Mills theory taken in a general axial gauge. So, for this part of the Lagrangian $\mathcal{L}(\boldsymbol{a}_\mu^\alpha, \boldsymbol{\psi})$ given by the zero-order terms in $1/M$, the spontaneous Lorentz violation again simply corresponds to a non-covariant gauge choice in an otherwise gauge invariant (and Lorentz invariant) theory. Furthermore one may expect that, as in the nonlinear QED model **?**, all the first and higher order terms in $1/M$ in $\mathcal{L}$ (2.25), though being by themselves Lorentz and CPT violating ones, do not cause physical SLIV effects due to the mutual cancellation of their contributions to the physical processes involved.

2.3 The lowest order SLIV processes

Let us now show that simple tree level calculations related to the Lagrangian $\mathcal{L}(\boldsymbol{a}_\mu^\alpha, \boldsymbol{\psi})$ confirm in essence this proposition. As an illustration, let's consider SLIV processes in the lowest order in g and $1/M$ being the fundamental parameters of the Lagrangian (2.25). They are, as one can readily see, the vector-fermion and vector-vector elastic scattering going in the order g/M which we are going to consider in some detail as soon as the Feynman rules in the Goldstonic Yang-Mills theory are established.

2.3.1 Feynman rules

The corresponding Feynman rules, apart from the ordinary Yang-Mills theory rules for

(i) the vector-fermion vertex

$$-ig\ \gamma_\mu\ t^i, \tag{2.26}$$

(ii) the vector field propagator (taken in a general axial gauge $n^\mu a_\mu^i = 0$)

$$D_{\mu\nu}^{ij}(k) = -\frac{i\delta^{ij}}{k^2}\left(g_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{n\cdot k} + \frac{n^2 k_\mu k_\nu}{(n\cdot k)^2}\right) \tag{2.27}$$

which automatically satisfies the orthogonality condition $n^\mu D_{\mu\nu}^{ij}(k) = 0$ and on-shell transversality $k_\mu D_{\mu\nu}^{ij}(k, k^2 = 0) = 0$ (the latter means that free vector fields with polarization vector $\epsilon_\mu^i(k, k^2 = 0)$ are always appeared transverse $k^\mu \epsilon_\mu^i(k) = 0$);

(iii) the 3-vector vertex (with vector field 4-momenta k_1, k_2 and k_3; all 4-momenta in vertexes are taken ingoing throughout)

$$gc^{ijk}[(k_1 - k_2)_\gamma g_{\alpha\beta} + (k_2 - k_3)_\alpha g_{\beta\gamma} + (k_3 - k_1)_\beta g_{\alpha\gamma}], \tag{2.28}$$

include the new ones, violating Lorentz and CPT invariance, for

(iv) the contact 2-vector-fermion vertex

$$i\frac{g\mathfrak{n}^2}{M}(\gamma\cdot\mathfrak{n}^k)\tau^k g_{\mu\nu}\ \delta^{ij}, \tag{2.29}$$

(v) another 3-vector vertex

$$-\frac{i\mathfrak{n}^2}{M}\left[(k_1\cdot\mathfrak{n}^i)k_{1,\alpha}g_{\beta\gamma}\delta^{jk}+(k_2\cdot\mathfrak{n}^j)k_{2,\beta}g_{\alpha\gamma}\delta^{ki}+(k_3\cdot\mathfrak{n}^k)k_{3,\gamma}g_{\alpha\beta}\delta^{ij}\right] \tag{2.30}$$

where the second index in the vector field 4-momenta k_1, k_2 and k_3 denotes their Lorentz components;

(vi) the extra 4-vector vertex (with the vector field 4-momenta $k_{1,2,3,4}$ and their proper differences $k_{12}\equiv k_1-k_2$ etc.)

$$\begin{aligned}&-\frac{\mathfrak{n}^2 g}{M}[c^{ijp}\delta^{kl}g_{\alpha\beta}g_{\gamma\delta}(\mathfrak{n}^p\cdot k_{12})+c^{klp}\delta^{ij}g_{\alpha\beta}g_{\gamma\delta}(\mathfrak{n}^p\cdot k_{34})+\\&+c^{ikp}\delta^{jl}g_{\alpha\gamma}g_{\beta\delta}(\mathfrak{n}^p\cdot k_{13})+c^{jlp}\delta^{ik}g_{\alpha\gamma}g_{\beta\delta}(\mathfrak{n}^p\cdot k_{24})+\\&+c^{ilp}\delta^{jk}g_{\alpha\delta}g_{\beta\gamma}(\mathfrak{n}^p\cdot k_{14})+c^{jkp}\delta^{il}g_{\alpha\delta}g_{\beta\gamma}(\mathfrak{n}^p\cdot k_{23})]\end{aligned} \tag{2.31}$$

where we have not included the terms which might contain contractions of the vacuum matrix $\mathfrak{n}^p_\mu$ with vector field polarization vectors $\epsilon^i_\mu(k)$ in the vector-vector scattering amplitude since these contractions are vanished due to the gauge taken (2.22), $\mathfrak{n}^p\cdot\epsilon^i=s^p(n\cdot\epsilon^i)=0$ (as follows according to a factorized two-vector form for the matrix $\mathfrak{n}^p_\mu$ (2.4)).

Just the rules (i-vi) are needed to calculate the lowest order processes mentioned in the above.

2.3.2 Vector boson scattering on fermion

This process is directly related to two SLIV diagrams one of which is given by the contact a^2-fermion vertex (2.29), while another corresponds to the pole diagram with the longitudinal a-boson exchange between Lorentz violating a^3 vertex (2.30) and ordinary a-boson-fermion one (2.26). Since ingoing and outgoing a-bosons appear transverse ($k_1\cdot\epsilon^i(k_1)=0$, $k_2\cdot\epsilon^j(k_2)=0$) only the third term in this a^3 coupling (2.30) contributes to the pole diagram so that one comes to a simple matrix element $i\mathcal{M}$ from both of diagrams

$$i\mathcal{M}=i\frac{gn^2}{M}\bar{u}(p_2)\tau^l\left[(\gamma\cdot\mathfrak{n}^l)+i(k\cdot\mathfrak{n}^l)\gamma^\mu k^\nu D_{\mu\nu}(k)\right]u(p_1)[\epsilon(k_1)\cdot\epsilon(k_2)] \tag{2.32}$$

where the spinors $u(p_{1,2})$ and polarization vectors $\epsilon_\mu^i(k_1)$ and $\epsilon_\mu^j(k_2)$ stand for ingoing and outgoing fermions and a-bosons, respectively, while k is the 4-momentum transfer $k = p_2 - p_1 = k_1 - k_2$. Upon further simplifications in the square bracket related to the explicit form of the a boson propagator $D_{\mu\nu}(k)$ (2.27) and matrix $\mathfrak{n}_\mu^i$ (2.4), and using the fermion current conservation $\bar{u}(p_2)(\hat{p}_2 - \hat{p}_1)u(p_1) = 0$, one is finally led to the total cancellation of the Lorentz violating contributions to the a-boson-fermion scattering in the g/M approximation.

Note, however, that such a result may be in some sense expected since from the SLIV point of view the lowest order a-boson-fermion scattering discussed here is hardly distinct from the photon-fermion scattering considered in the nonlinear QED case**?**. Actually, the fermion current conservation which happens to be crucial for the above cancellation works in both of cases, whereas the couplings which are peculiar to the Yang-Mills theory have not yet touched on. In this connection the next example seems to be more instructive.

2.3.3 Vector-vector scattering

The matrix element for this process in the lowest order g/M is given by the contact SLIV a^4 vertex (2.31) and the pole diagrams with the longitudinal a-boson exchange between the ordinary a^3 vertex (2.28) and Lorentz violating a^3 one (2.30), and vice versa. There are six pole diagrams in total describing the elastic $a - a$ scattering in the s- and t-channels, respectively, including also those with an interchange of identical a-bosons. Remarkably, the contribution of each of them is exactly canceled with one of six terms appeared in the contact vertex (2.31). Actually, writing down the matrix element for one of the pole diagrams with ingoing a-bosons (with momenta k_1 and k_2) interacting through the vertex (2.28) and outgoing a-bosons (with momenta k_3 and k_4) interacting through the vertex (2.30) one has

$$\begin{aligned} i\mathcal{M}_{pole}^{(1)} &= -i\frac{gn^2}{M}c^{ijp}\delta^{kl}[(k_1-k_2)_\mu g_{\alpha\beta} + (k_2-k)_\alpha g_{\beta\mu} + (k-k_1)_\beta g_{\alpha\mu}] \cdot \\ &\quad \cdot D_{\mu\nu}^{pq}(k) g_{\gamma\delta} k_\nu (n^q \cdot k)[\epsilon^{i,\alpha}(k_1)\epsilon^{j,\beta}(k_2)\epsilon^{k,\gamma}(k_3)\epsilon^{l,\delta}(k_4)] \end{aligned} \tag{2.33}$$

where polarization vectors $\epsilon^{i,\alpha}(k_1)$, $\epsilon^{j,\beta}(k_2)$, $\epsilon^{k,\gamma}(k_3)$ and $\epsilon^{l,\delta}(k_4)$ belong, respectively, to ingoing and outgoing a-bosons, while $k = -(k_1+k_2) = k_3+k_4$ according to the momentum running in the diagrams taken above. Again, as in the previous case of vector-fermion scattering, due to the fact that outgoing a-bosons appear transverse ($k_3 \cdot \epsilon^k(k_3) = 0$ and $k_4 \cdot \epsilon^l(k_4) = 0$), only the third term in the Lorentz violating a^3 coupling (2.30) contributes to this pole diagram. After evident simplifications related to the a-boson propagator $D_{\mu\nu}(k)$ (2.27) and matrix $\mathfrak{n}^i_\mu$ (2.4) one comes to the expression which is exactly cancelled with the first term in the contact SLIV vertex (2.31) when it is properly contracted with a-boson polarization vectors. Likewise, other terms in this vertex provide the further one-to-one cancellation with the remaining pole matrix elements $i\mathcal{M}^{(2-6)}_{pole}$. So, again, the Lorentz violating contribution to the vector-vector scattering is absent in Goldstonic Yang-Mills theory in the lowest g/M approximation.

2.3.4 Other processes

Many other tree level Lorentz violating processes, related to a bosons and fermions, appear in higher orders in the basic SLIV parameter $1/M$. They come from the subsequent expansion of the effective Higgs mode (2.21) in the Lagrangian (2.25). Again, their amplitudes are essentially determined by an interrelation between the longitudinal a-boson exchange diagrams and the corresponding contact a-boson interaction diagrams which appear to cancel each other thus eliminating physical Lorentz violation in theory.

Most likely, the same conclusion can be derived for SLIV loop contributions as well. Actually, as in the massless QED case considered earlier Azatov and Chkareuli (2006), the corresponding one-loop matrix elements in Goldstonic Yang-Mills theory either vanish by themselves or amount to the differences between pairs of the similar integrals whose integration variables are shifted relative to each other by some constants (being in general arbitrary functions of external four-momenta of the particles involved) that in the framework of dimensional regularization leads to their total cancellation.

So, the Goldstonic vector field theory (2.25) for a non-Abelian charge-carrying matter is likely to be physically indistinguishable from a conventional Yang-Mills theory.

Chapter 3

Spontaneously Generated Gravitons

In this chapter is used a similar nonlinear constraint for a symmetric two-index tensor field

$$H_{\mu\nu}^2 = \mathfrak{n}^2 M^2 \ , \qquad \mathfrak{n}^2 \equiv \mathfrak{n}_{\mu\nu}\mathfrak{n}^{\mu\nu} = \pm 1 \tag{3.1}$$

(where $\mathfrak{n}_{\mu\nu}$ is now a properly oriented 'unit' Lorentz tensor, while M is the proposed scale for Lorentz violation) which fixes its length in a similar way to the vector field case above. Also, in analogy to the nonlinear QED case (?) with its gauge invariant Lagrangian, we propose the linearized Einstein-Hilbert kinetic term for the tensor field, which by itself preserves a diffeomorphism invariance. One will see that such a SLIV pattern (3.1), due to which the true vacuum in the theory is chosen, induces massless tensor Goldstone modes some of which can naturally be collected in the physical graviton. The linearized theory we start with becomes essentially nonlinear, when expressed in terms of the pure Goldstone modes, and contains a variety of Lorentz (and CPT) violating couplings. However, all SLIV effects turn out to be strictly cancelled in physical processes once the tensor field gravity theory (being considered as the weak-field limit of general relativity (GR)) is properly extended to GR. So, this formulation of SLIV seems to amount to the fixing of a gauge for the tensor field in a special manner making the Lorentz violation only superficial just as in the nonlinear QED framework (?). From this viewpoint, both conventional QED and GR theories appear to be generic Goldstonic theories in which some of the gauge degrees of freedom of these fields are condensed (thus eventually emerging as a non-covariant gauge choice), while their massless NG modes are collected in photons or gravitons in such a way that the physical Lorentz invariance is ultimately preserved. However, there might appear some principal distinctions between conventional and Goldstonic theories if, as we argue later, the underlying local symmetry were slightly broken at very small distances in a way that could eventually allow us to differentiate between them in an observational way.

The chapter mostly is founded on my publication (Chkareuli et al., 2011) and is organized in the following way. In section 3.1 we formulate the model for tensor field gravity and find massless NG modes some of which are collected in the physical graviton. Then in section 3.2 we derive general Feynman rules for the basic graviton-graviton and graviton-matter (scalar) field interactions in the Goldstonic gravity theory. In essence the model contains two perturbative parameters, the inverse Planck and SLIV mass scales, $1/M_P$ and $1/M$, respectively, so that the SLIV interactions are always proportional to some powers of them. Some lowest order SLIV processes, such as graviton-graviton scattering and graviton scattering off the massive scalar field, are considered in detail. We show that all these Lorentz violating effects, taken in the tree approximation, in fact turn out to vanish so that physical Lorentz invariance is ultimately restored.

3.1 The Model

According to our philosophy, we propose to consider the tensor field gravity theory which mimics linearized general relativity in Minkowski space-time. The corresponding Lagrangian for one real scalar field ϕ (representing all sorts of matter in the model)

$$\mathcal{L}(H_{\mu\nu},\phi)=\mathcal{L}(H)+\mathcal{L}(\phi)+\mathcal{L}_{int} \tag{3.2}$$

consists of the tensor field kinetic terms of the form

$$\mathcal{L}(H)=\frac{1}{2}\partial_{\lambda}H^{\mu\nu}\partial^{\lambda}H_{\mu\nu}-\frac{1}{2}\partial_{\lambda}H_{tr}\partial^{\lambda}H_{tr}-\partial_{\lambda}H^{\lambda\nu}\partial^{\mu}H_{\mu\nu}+\partial^{\nu}H_{tr}\partial^{\mu}H_{\mu\nu}\ , \tag{3.3}$$

(H_{tr} stands for the trace of $H_{\mu\nu}$, $H_{tr}=\eta^{\mu\nu}H_{\mu\nu}$) which is invariant under the diff transformations

$$\delta H_{\mu\nu}=\partial_{\mu}\xi_{\nu}+\partial_{\nu}\xi_{\mu}\ ,\qquad \delta x^{\mu}=\xi^{\mu}(x)\ , \tag{3.4}$$

together with the free scalar field and interaction terms

$$\mathcal{L}(\phi)=\frac{1}{2}\left(\partial_{\rho}\phi\partial^{\rho}\phi-m^{2}\phi^{2}\right)\ ,\qquad \mathcal{L}_{int}=\frac{1}{M_{P}}H_{\mu\nu}T^{\mu\nu}(\phi)\ . \tag{3.5}$$

Here $T^{\mu\nu}(\phi)$ is the conventional energy-momentum tensor for a scalar field

$$T^{\mu\nu}(\phi)=\partial^{\mu}\phi\partial^{\nu}\phi-\eta^{\mu\nu}\mathcal{L}(\phi)\ , \tag{3.6}$$

and the coupling constant in $\mathcal{L}_{int}$ is chosen to be the inverse of the Planck mass M_P. It is clear that, in contrast to the tensor field kinetic terms, the other terms in (3.2) are only approximately invariant under the diff transformations (3.4), as they correspond to the weak-field limit in GR. Following the nonlinear σ-model for QED (?), we propose the SLIV condition (3.1) as some tensor field length-fixing constraint which is supposed to be substituted into the total Lagrangian $\mathcal{L}(H_{\mu\nu},\phi)$ prior to the variation of the action. This eliminates, as will be seen, a massive Higgs mode in the final theory thus leaving only massless Goldstone modes, some of which are then collected in the physical graviton.

Let us first turn to the spontaneous Lorentz violation itself, which is caused by the constraint (3.1). This constraint can be written in the more explicit form

$$H_{\mu\nu}^2 = H_{00}^2 + H_{i=j}^2 + (\sqrt{2}H_{i\neq j})^2 - (\sqrt{2}H_{0i})^2 = \mathfrak{n}^2 M^2 = \pm\, M^2 \tag{3.7}$$

(where the summation over all indices $(i, j = 1, 2, 3)$ is imposed) and means in essence that the tensor field $H_{\mu\nu}$ develops the vacuum expectation value (vev) configuration

$$< H_{\mu\nu}(x) > \;\; = \mathfrak{n}_{\mu\nu} M \tag{3.8}$$

determined by the matrix $\mathfrak{n}_{\mu\nu}$. The initial Lorentz symmetry $SO(1,3)$ of the Lagrangian $\mathcal{L}(H_{\mu\nu},\phi)$ given in (3.2) then formally breaks down at a scale M to one of its subgroups. If one assumes a "minimal" vacuum configuration in the $SO(1,3)$ space with the vev (3.8) developed on a single $H_{\mu\nu}$ component, there are in fact the following three possibilities

$$\begin{aligned}
(a) \quad & \mathfrak{n}_{00} \neq 0\,, \quad SO(1,3) \to SO(3) \\
(b) \quad & \mathfrak{n}_{i=j} \neq 0\,, \quad SO(1,3) \to SO(1,2) \\
(c) \quad & \mathfrak{n}_{i\neq j} \neq 0\,, \quad SO(1,3) \to SO(1,1)
\end{aligned} \tag{3.9}$$

for the positive sign in (3.7), and

$$(d) \quad \mathfrak{n}_{0i} \neq 0\,, \quad SO(1,3) \to SO(2) \tag{3.10}$$

for the negative sign. These breaking channels can be readily derived by counting how many different eigenvalues the vev matrix $\mathfrak{n}$ has for each particular case (*a*-*d*). Accordingly, there are only three Goldstone modes in the cases (a, b) and five modes in the cases $(c\text{-}d)$ [1]. In order to associate at least one of the two transverse polarization states of the physical graviton with these modes, one could have any of the above-mentioned SLIV channels except for the case (a). Indeed, it is impossible for the graviton to have all vanishing spatial components, as one needs for the Goldstone modes in case (a). Therefore, no linear combination of the three Goldstone modes in case (a) could behave like the physical graviton (see more detailed consideration in Carroll et al. (2009)). Apart from the minimal vev configuration, there are many others as well. A particular case of interest is that of the traceless vev tensor $\mathfrak{n}_{\mu\nu}$

$$\mathfrak{n}_{\mu\nu}\eta^{\mu\nu} = 0 \tag{3.11}$$

in terms of which the Goldstonic gravity Lagrangian acquires an especially simple form (see below). It is clear that the vev in this case can be developed on several $H_{\mu\nu}$ components simultaneously, which in general may lead to total Lorentz violation with all six Goldstone modes generated. For simplicity, we will use this form of vacuum configuration in what follows, while our arguments can be applied to any type of vev tensor $\mathfrak{n}_{\mu\nu}$.

Aside from the pure Lorentz Goldstone modes, the question of the other components of the symmetric two-index tensor $H_{\mu\nu}$ naturally arises. Remarkably, they turn out to be Pseudo Goldstone modes (PGMs) in the theory. Indeed, although we only propose Lorentz invariance of the Lagrangian $\mathcal{L}(H_{\mu\nu}, \phi)$, the SLIV constraint (3.1) formally possesses the much higher accidental symmetry $SO(7, 3)$ of the constrained bilinear form (3.7), which manifests itself when considering the $H_{\mu\nu}$ components as the "vector" ones under $SO(7, 3)$.

[1] Indeed, the vev matrices in the cases (a, b) look, respectively, as $\mathfrak{n}^{(a)} = diag(1, 0, 0, 0)$ and $\mathfrak{n}^{(b)} = diag(0, 1, 0, 0)$, while in the cases $(c\text{-}d)$ these matrices, taken in the diagonal bases, have the forms $\mathfrak{n}^{(c)} = diag(0, 1, -1, 0)$ and $\mathfrak{n}^{(d)} = diag(1, -1, 0, 0)$, respectively (for certainty, we fixed $i = j = 1$ in the case (b), $i = 1$ and $j = 2$ in the case (c), and $i = 1$ in the case (d)). The groups of invariance of these vev matrices are just the surviving Lorentz subgroups indicated on the right-handed sides in (3.9) and (3.10). The broken Lorentz generators determine then the numbers of Goldstone modes mentioned above.

This symmetry is in fact spontaneously broken, side by side with Lorentz symmetry, at the scale M. Assuming again a minimal vacuum configuration in the $SO(7,3)$ space, with the vev (3.8) developed on a single $H_{\mu\nu}$ component, we have either time-like ($SO(7,3) \to SO(6,3)$) or space-like ($SO(7,3) \to SO(7,2)$) violations of the accidental symmetry depending on the sign of $\mathfrak{n}^2 = \pm 1$ in (3.7). According to the number of broken $SO(7,3)$ generators, just nine massless NG modes appear in both cases. Together with an effective Higgs component, on which the vev is developed, they complete the whole ten-component symmetric tensor field $H_{\mu\nu}$ of the basic Lorentz group. Some of them are true Goldstone modes of the spontaneous Lorentz violation, others are PGMs since, as was mentioned, an accidental $SO(7,3)$ symmetry is not shared by the whole Lagrangian $\mathcal{L}(H_{\mu\nu}, \phi)$ given in (3.2). Notably, in contrast to the scalar PGM case Weinberg, they remain strictly massless being protected by the starting diff invariance[2] which becomes exact when the tensor field gravity Lagrangian (3.2) is properly extended to GR. Owing to this invariance, some of the Lorentz Goldstone modes and PGMs can then be gauged away from the theory, as usual.

Now, one can rewrite the Lagrangian $\mathcal{L}(H_{\mu\nu}, \phi)$ in terms of the Goldstone modes explicitly using the SLIV constraint (3.1). For this purpose, let us take the following handy parameterization for the tensor field $H_{\mu\nu}$

$$H_{\mu\nu} = h_{\mu\nu} + \frac{n_{\mu\nu}}{n^2}(\mathfrak{n} \cdot H) \qquad (\mathfrak{n} \cdot H \equiv \mathfrak{n}_{\mu\nu} H^{\mu\nu}) \tag{3.12}$$

where $h_{\mu\nu}$ corresponds to the pure Goldstonic modes [3] satisfying

[2]For non-minimal vacuum configuration when vevs are developed on several $H_{\mu\nu}$ components, thus leading to a more substantial breaking of the accidental $SO(7,3)$ symmetry, some extra PGMs are also generated. However, they are not protected by a diff invariance and acquire masses of the order of the breaking scale M.

[3]It should be particularly emphasized that the modes collected in the $h_{\mu\nu}$ are in fact the Goldstone modes of the broken accidental $SO(7,3)$ symmetry of the constraint (3.1), thus containing the Lorentz Goldstone modes and PGMs put together.

$$\mathfrak{n}\cdot h = 0 \qquad (\mathfrak{n}\cdot h \equiv \mathfrak{n}_{\mu\nu}h^{\mu\nu}) \tag{3.13}$$

while the effective "Higgs" mode (or the $H_{\mu\nu}$ component in the vacuum direction) is given by the scalar product $\mathfrak{n}\cdot H$. Substituting this parameterization (3.12) into the tensor field constraint (3.1), one comes to the equation for $\mathfrak{n}\cdot H$

$$\mathfrak{n}\cdot H \; = (M^2 - \mathfrak{n}^2 h^2)^{\frac{1}{2}} = M - \frac{\mathfrak{n}^2 h^2}{2M} + O(1/M^2) \tag{3.14}$$

taking, for definiteness, the positive sign for the square root and expanding it in powers of h^2/M^2, $h^2 \equiv h_{\mu\nu}h^{\mu\nu}$. Putting then the parameterization (3.12) with the SLIV constraint (3.14) into the Lagrangian $\mathcal{L}(H_{\mu\nu},\phi)$ given in (3.2, 3.3, 3.5), one comes to the truly Goldstonic tensor field gravity Lagrangian $\mathcal{L}(h_{\mu\nu},\phi)$ containing an infinite series in powers of the $h_{\mu\nu}$ modes. For the traceless vev tensor $\mathfrak{n}_{\mu\nu}$ (3.11) it takes, without loss of generality, the especially simple form

$$\begin{aligned}\mathcal{L}(h_{\mu\nu},\phi) \;=\; & \frac{1}{2}\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu} - \frac{1}{2}\partial_\lambda h_{tr}\partial^\lambda h_{tr} - \partial_\lambda h^{\lambda\nu}\partial^\mu h_{\mu\nu} + \partial^\nu h_{tr}\partial^\mu h_{\mu\nu} + \\ & + \frac{1}{2M}h^2\left[-2\mathfrak{n}^{\mu\lambda}\partial_\lambda\partial^\nu h_{\mu\nu} + \mathfrak{n}^2(\mathfrak{n}\partial\partial)h_{tr}\right] + \frac{1}{8M^2}h^2\left[-\mathfrak{n}^2\partial^2 + 2(\partial\mathfrak{n}\mathfrak{n}\partial)\right]h^2 \\ & + \mathcal{L}(\phi) + \frac{M}{M_P}\mathfrak{n}^2\left[\mathfrak{n}_{\mu\nu}\partial^\mu\phi\partial^\nu\phi\right] + \frac{1}{M_P}h_{\mu\nu}T^{\mu\nu} + \frac{1}{2MM_P}h^2\left[-\mathfrak{n}_{\mu\nu}\partial^\mu\phi\partial^\nu\phi\right]\end{aligned} \tag{3.15}$$

written in the $O(h^2/M^2)$ approximation in which, besides the conventional graviton bilinear kinetic terms, there are also three- and four-linear interaction terms in powers of $h_{\mu\nu}$ in the Lagrangian. Some of the notations used are collected below

$$\begin{aligned} h^2 \;&\equiv\; h_{\mu\nu}h^{\mu\nu}\ , \quad h_{tr} \equiv \eta^{\mu\nu}h_{\mu\nu}\ , \\ \mathfrak{n}\partial\partial \;&\equiv\; \mathfrak{n}_{\mu\nu}\partial^\mu\partial^\nu\ , \quad \partial\mathfrak{n}\mathfrak{n}\partial \equiv \partial^\mu\mathfrak{n}_{\mu\nu}\mathfrak{n}^{\nu\lambda}\partial_\lambda\ . \end{aligned} \tag{3.16}$$

The bilinear scalar field term

$$\frac{M}{M_P}\mathfrak{n}^2\left[\mathfrak{n}_{\mu\nu}\partial^\mu\phi\partial^\nu\phi\right] \tag{3.17}$$

in the third line in the Lagrangian (3.15) merits special notice. This term arises from the interaction Lagrangian $\mathcal{L}_{int}$ (3.5) after application of the tracelessness condition (3.11) for the vev tensor $\mathfrak{n}_{\mu\nu}$. It could significantly affect the dispersion relation for the scalar field ϕ (and any other sort of matter as well) thus leading to an unacceptably large Lorentz violation if the SLIV scale M were comparable with the Planck mass M_P. However, this term can be gauged away by an appropriate redefinition (going to new coordinates $x^\mu \to x^\mu + \xi^\mu$) of the scalar field derivative according to

$$\partial^\mu\phi \to \partial^\mu\phi + \partial_\rho\xi^\mu\partial^\rho\phi \tag{3.18}$$

In fact with the following choice of the parameter function $\xi^\mu(x)$

$$\xi^\mu(x) = \frac{M}{2M_P}\mathfrak{n}^2\mathfrak{n}^{\mu\nu}x_\nu \ ,$$

the term (3.17) is cancelled by an analogous term stemming from the scalar field kinetic term in $\mathcal{L}(\phi)$ given in (3.5) [4]. On the other hand, since the diff invariance is an approximate symmetry of the Lagrangian $\mathcal{L}(H_{\mu\nu},\phi)$ we started with (3.2), this cancellation will only be accurate up to the linear order corresponding to the tensor field theory. Indeed, a proper extension of this theory to GR with its exact diff invariance will ultimately restore the usual dispersion relation for the scalar (and other matter) fields. Taking this into account, we will henceforth omit the term (3.17) in $\mathcal{L}(h_{\mu\nu},\phi)$ thus keeping the "normal" dispersion relation for the scalar field in what follows.

Together with the Lagrangian one must also specify other supplementary conditions for the tensor field $h^{\mu\nu}$(appearing eventually as possible gauge fixing terms in the Goldstonic

[4]In the general case, with the vev tensor $\mathfrak{n}_{\mu\nu}$ having a non-zero trace, this cancellation would also require the redefinition of the scalar field itself as $\phi \to \phi(1-\mathfrak{n}_{\mu\nu}\eta^{\mu\nu}\frac{M}{M_P})^{-1/2}$.

tensor field gravity) in addition to the basic Goldstonic "gauge" condition $\mathfrak{n}_{\mu\nu}h^{\mu\nu} = 0$ given above (3.13). The point is that the spin 1 states are still left in the theory and are described by some of the components of the new tensor $h_{\mu\nu}$. This is certainly inadmissible [5]. Usually, the spin 1 states (and one of the spin 0 states) are excluded by the conventional Hilbert-Lorentz condition

$$\partial^{\mu}h_{\mu\nu} + q\partial_{\nu}h_{tr} = 0 \tag{3.19}$$

(q is an arbitrary constant, giving for $q = -1/2$ the standard harmonic gauge condition). However, as we have already imposed the constraint (3.13), we can not use the full Hilbert-Lorentz condition (3.19) eliminating four more degrees of freedom in $h_{\mu\nu}$. Otherwise, we would have an "over-gauged" theory with a non-propagating graviton. In fact, the simplest set of conditions which conform with the Goldstonic condition (3.13) turns out to be

$$\partial^{\rho}(\partial_{\mu}h_{\nu\rho} - \partial_{\nu}h_{\mu\rho}) = 0 \tag{3.20}$$

This set excludes only three degrees of freedom [6] in $h_{\mu\nu}$ and, besides, it automatically satisfies the Hilbert-Lorentz spin condition as well. So, with the Lagrangian (3.15) and the supplementary conditions (3.13) and (3.20) lumped together, one eventually comes to a working model for the Goldstonic tensor field gravity. Generally, from ten components of the symmetric two-index tensor $h_{\mu\nu}$ four components are excluded by the supplementary conditions (3.13) and (3.20). For a plane gravitational wave propagating in, say, the z direction another four components are also eliminated, due to the fact that the above supplementary conditions still leave freedom in the choice of a coordinate system, $x^{\mu} \to x^{\mu} + \xi^{\mu}(t - z/c)$, much as it

[5] Indeed, spin 1 must be necessarily excluded as the sign of the energy for spin 1 is always opposite to that for spin 2 and 0.

[6] The solution for a gauge function $\xi_{\mu}(x)$ satisfying the condition (3.20) can generally be chosen as $\xi_{\mu} = \Box^{-1}(\partial^{\rho}h_{\mu\rho}) + \partial_{\mu}\theta$, where $\theta(x)$ is an arbitrary scalar function, so that only three degrees of freedom in $h_{\mu\nu}$ are actually eliminated.

takes place in standard GR. Depending on the form of the vev tensor $\mathfrak{n}_{\mu\nu}$, caused by SLIV, the two remaining transverse modes of the physical graviton may consist solely of Lorentz Goldstone modes or of Pseudo Goldstone modes, or include both of them.

3.2 The Lowest Order SLIV Processes

The Goldstonic gravity Lagrangian (3.15) looks essentially nonlinear and contains a variety of Lorentz and CPT violating couplings when expressed in terms of the pure tensor Goldstone modes. However, as we show below, all violation effects turn out to be strictly cancelled in the lowest order SLIV processes. Such a cancellation in vector-field theories, both Abelian (**?**Azatov and Chkareuli, 2006; Chkareuli and Kepuladze, 2007) and non-Abelian (Chkareuli and Jejelava, 2008), and, therefore, their equivalence to conventional QED and Yang-Mills theories, allows one to conclude that the nonlinear SLIV constraint in these theories amounts to a non-covariant gauge choice in an otherwise gauge invariant and Lorentz invariant theory. It seems that a similar conclusion can be made for tensor field gravity, i.e. the SLIV constraint (3.1) corresponds to a special gauge choice in a diff and Lorentz invariant theory. This conclusion certainly works for the diff invariant free tensor field part (3.3) in the starting Lagrangian $\mathcal{L}(H_{\mu\nu}, \phi)$. On the other hand, its matter field sector (3.5), possessing only an approximate diff invariance, might lead to an actual Lorentz violation through the deformed dispersion relations of the matter fields involved. However, as was mentioned above, a proper extension of the tensor field theory to GR with its exact diff invariance ultimately restores the dispersion relations for matter fields and, therefore, the SLIV effects vanish. Taking this into account, we omit the term (3.17) in the Goldstonic gravity Lagrangian $\mathcal{L}(h_{\mu\nu}, \phi)$ thus keeping the "normal" dispersion relation for the scalar field representing all the matter in our model.

We are now going to consider the lowest order SLIV processes, after first establishing the Feynman rules in the Goldstonic gravity theory. We use for simplicity, both in the Lagrangian $\mathcal{L}$ (3.15) and forthcoming calculations, the traceless vev tensor $\mathfrak{n}_{\mu\nu}$, while our results remain true for any type of vacuum configuration caused by SLIV.

3.2.1 Feynman rules

The Feynman rules stemming from the Lagrangian $\mathcal{L}$ (3.15) for the pure graviton sector are as follows:

(**i**) The first and most important is the graviton propagator which only conforms with the Lagrangian (3.15) and the gauge conditions (3.13) and (3.20)

$$\begin{aligned}
-iD_{\mu\nu\alpha\beta}(k) = & \frac{1}{2k^2}\left(\eta_{\beta\mu}\eta_{\alpha\nu}+\eta_{\beta\nu}\eta_{\alpha\mu}-\eta_{\alpha\beta}\eta_{\mu\nu}\right) \\
& -\frac{1}{2k^4}\left(\eta_{\beta\nu}k_\alpha k_\mu+\eta_{\alpha\nu}k_\beta k_\mu+\eta_{\beta\mu}k_\alpha k_\nu+\eta_{\alpha\mu}k_\beta k_\nu\right) \\
& -\frac{1}{k^2(\mathfrak{n}kk)}\left(k_\alpha k_\beta \mathfrak{n}_{\mu\nu}+k_\nu k_\mu \mathfrak{n}_{\alpha\beta}\right)+\frac{1}{k^2(\mathfrak{n}kk)^2}\left[\mathfrak{n}^2-\frac{2}{k^2}(k\mathfrak{n}\mathfrak{n}k)\right]k_\mu k_\nu k_\alpha k_\beta \\
& +\frac{1}{k^4(\mathfrak{n}kk)}\left(\mathfrak{n}_{\mu\rho}k^\rho k_\nu k_\alpha k_\beta+\mathfrak{n}_{\nu\rho}k^\rho k_\mu k_\alpha k_\beta+\mathfrak{n}_{\alpha\rho}k^\rho k_\nu k_\mu k_\beta+\mathfrak{n}_{\beta\rho}k^\rho k_\nu k_\alpha k_\mu\right)
\end{aligned} \tag{3.21}$$

(where $(\mathfrak{n}kk)\equiv\mathfrak{n}_{\mu\nu}k^\mu k^\nu$ and $(k\mathfrak{n}\mathfrak{n}k)\equiv k^\mu\mathfrak{n}_{\mu\nu}\mathfrak{n}^{\nu\lambda}k_\lambda$). It automatically satisfies the orthogonality condition $\mathfrak{n}^{\mu\nu}D_{\mu\nu\alpha\beta}(k)=0$ and on-shell transversality $k^\mu k^\nu D_{\mu\nu\alpha\beta}(k,k^2=0)=0$. This is consistent with the corresponding polarization tensor $\epsilon_{\mu\nu}(k,k^2=0)$ of the free tensor fields, being symmetric, traceless ($\eta^{\mu\nu}\epsilon_{\mu\nu}=0$), transverse ($k^\mu\epsilon_{\mu\nu}=0$), and also orthogonal to the vacuum direction, $\mathfrak{n}^{\mu\nu}\epsilon_{\mu\nu}(k)=0$. Apart from that, the gauge invariance allows us to write the polarization tensor in the factorized form[7] , $\epsilon_{\mu\nu}(k)=\epsilon_\mu(k)\epsilon_\nu(k)$, and to proceed with the above-mentioned tracelessness and transversality expressed as the simple conditions $\epsilon_\mu\epsilon^\mu=0$ and $k^\mu\epsilon_\mu=0$ respectively. In the following we will use these simplifications. As one can see, only the standard terms given by the first bracket in (3.21) contribute when the propagator is sandwiched between conserved energy-momentum tensors of matter fields, and the result is always Lorentz invariant.

(**ii**) Next is the 3-graviton vertex with graviton polarization tensors (and 4-momenta) given by $\epsilon^{\alpha\alpha'}(k_1)$, $\epsilon^{\beta\beta'}(k_2)$ and $\epsilon^{\gamma\gamma'}(k_3)$

[7] Weinberg (1964b,a, 1965); Gross and Jackiw (1968)

$$
\begin{aligned}
&-\frac{i}{2M}P^{\alpha\alpha'}(k_1)\left(\eta^{\beta\gamma}\eta^{\beta'\gamma'}+\eta^{\beta\gamma'}\eta^{\beta'\gamma}\right)\\
&-\frac{i}{2M}P^{\beta\beta'}(k_2)\left(\eta^{\alpha\gamma}\eta^{\alpha'\gamma'}+\eta^{\alpha\gamma'}\eta^{\alpha'\gamma}\right)\\
&-\frac{i}{2M}P^{\gamma\gamma'}(k_3)\left(\eta^{\beta\alpha}\eta^{\beta'\alpha'}+\eta^{\beta\alpha'}\eta^{\beta'\alpha}\right)
\end{aligned}
\tag{3.22}
$$

where the momentum tensor $P^{\mu\nu}(k)$ is

$$
P^{\mu\nu}(k)=-\mathfrak{n}^{\nu\rho}k_\rho k^\mu-\mathfrak{n}^{\mu\rho}k_\rho k^\nu+\eta^{\mu\nu}\mathfrak{n}^{\rho\sigma}k_\rho k_\sigma\ . \tag{3.23}
$$

Note that all 4-momenta at the vertices are taken ingoing throughout.

(**iii**) Finally, the 4-graviton vertex with the graviton polarization tensors (and 4-momenta) $\epsilon^{\alpha\alpha'}(k_1)$, $\epsilon^{\beta\beta'}(k_2)$, $\epsilon^{\gamma\gamma'}(k_3)$ and $\epsilon^{\delta\delta'}(k_4)$

$$
\begin{aligned}
&iQ_{\mu\nu}\left(\eta^{\alpha\beta}\eta^{\alpha'\beta'}+\eta^{\alpha\beta'}\eta^{\alpha'\beta}\right)\left(\eta^{\gamma\delta}\eta^{\gamma'\delta'}+\eta^{\gamma\delta'}\eta^{\gamma'\delta}\right)(k_1+k_2)^\mu(k_1+k_2)^\nu\\
&+iQ_{\mu\nu}\left(\eta^{\alpha\gamma}\eta^{\alpha'\gamma'}+\eta^{\alpha\gamma'}\eta^{\alpha'\gamma}\right)\left(\eta^{\beta\delta}\eta^{\beta'\delta'}+\eta^{\beta\delta'}\eta^{\beta'\delta}\right)(k_1+k_3)^\mu(k_1+k_3)^\nu\\
&+iQ_{\mu\nu}\left(\eta^{\alpha\delta}\eta^{\alpha'\delta'}+\eta^{\alpha\delta'}\eta^{\alpha'\delta}\right)\left(\eta^{\gamma\beta}\eta^{\gamma'\beta'}+\eta^{\gamma\beta'}\eta^{\gamma'\beta}\right)(k_1+k_4)^\mu(k_1+k_4)^\nu.
\end{aligned}
\tag{3.24}
$$

Here we have used the self-evident identities for all ingoing momenta ($k_1+k_2+k_3+k_4=0$), such as

$$
(k_1+k_2)^\mu(k_1+k_2)^\nu+(k_3+k_4)^\mu(k_3+k_4)^\nu=2(k_1+k_2)^\mu(k_1+k_2)^\nu
$$

and so on, and denoted by $Q_{\mu\nu}$ the expression

$$
Q_{\mu\nu}\equiv-\frac{1}{4M^2}(-\mathfrak{n}^2\eta_{\mu\nu}+2\mathfrak{n}_{\mu\rho}\mathfrak{n}_\nu^\rho)\ . \tag{3.25}
$$

Coming now to the gravitational interaction of the scalar field, one has two more vertices:

(**iv**) The standard graviton-scalar-scalar vertex with the graviton polarization tensor $\epsilon^{\alpha\alpha'}$ and the scalar field 4-momenta p_1 and p_2

$$-\frac{i}{M_p}\left(p_1^{\alpha}p_2^{\alpha'}+p_2^{\alpha}p_1^{\alpha'}\right)+\frac{i}{M_p}\eta^{\alpha\alpha'}[(p_1p_2)+m^2] \tag{3.26}$$

where (p_1p_2) stands for the scalar product.

(**v**) The contact graviton-graviton-scalar-scalar interaction caused by SLIV with the graviton polarization tensors $\epsilon^{\alpha\alpha'}$ and $\epsilon^{\beta\beta'}$ and the scalar field 4-momenta p_1 and p_2

$$\frac{i}{MM_p}\left(g^{\alpha\beta}g^{\alpha'\beta'}+g^{\alpha\beta'}g^{\alpha'\beta}\right)\left(\mathfrak{n}_{\mu\nu}p_1^{\mu}p_2^{\nu}\right)\ . \tag{3.27}$$

Just the rules (**i-v**) are needed to calculate the lowest order processes mentioned above.

3.2.2 Graviton-graviton scattering

The matrix element for this SLIV process to the lowest order $1/M^2$ is given by the contact h^4 vertex (3.24) and the pole diagrams with longitudinal graviton exchange between two Lorentz violating h^3 vertices (3.22). There are three pole diagrams in total, describing the elastic graviton-graviton scattering in the s- and t-channels respectively, and also the diagram with an interchange of identical gravitons. Remarkably, the contribution of each of them is exactly cancelled by one of three terms appearing in the contact vertex (3.24). Actually, for the s-channel pole diagrams with ingoing gravitons with polarizations (and 4-momenta) $\epsilon_1(k_1)$ and $\epsilon_2(k_2)$ and outgoing gravitons with polarizations (and 4-momenta) $\epsilon_3(k_3)$ and $\epsilon_4(k_4)$ one has, after some evident simplifications related to the graviton propagator $D_{\mu\nu}(k)$ (3.21) inside the matrix element

$$i\mathcal{M}_{pole}^{(1)}=i\frac{1}{M^2}(\epsilon_1\cdot\epsilon_2)^2(\epsilon_3\cdot\epsilon_4)^2(-\mathfrak{n}^2k^2+2k^{\mu}\mathfrak{n}_{\mu\nu}\mathfrak{n}^{\nu\lambda}k_{\lambda}). \tag{3.28}$$

Here $k=k_1+k_2=-(k_3+k_4)$ is the momentum running in the diagrams listed above, and all the polarization tensors are properly factorized throughout, $\epsilon_{\mu\nu}(k)=\epsilon_{\mu}(k)\epsilon_{\nu}(k)$, as was mentioned above. We have also used that, since ingoing and outgoing gravitons appear transverse ($k_a^\mu \epsilon_\mu(k_a) = 0$, $a = 1, 2, 3, 4$), only the third term in the momentum tensors $P^{\mu\nu}(k_a)$ (3.23) in the h^3 couplings (3.22) contributes to all pole diagrams. Now, one can readily confirm that this matrix element is exactly cancelled with the first term in the contact SLIV vertex (3.24), when it is properly contracted with the graviton polarization vectors. In a similar manner, two other terms in the contact vertex provide the further one-to-one cancellations with the remaining two pole matrix elements $i\mathcal{M}_{pole}^{(2,3)}$. So, the Lorentz violating contribution to graviton-graviton scattering is absent in Goldstonic gravity theory in the lowest $1/M^2$ approximation.

3.2.3 Graviton scattering on a massive scalar

This SLIV process appears in the order $1/MM_p$ (in contrast to the conventional $1/M_p^2$ order graviton-scalar scattering). It is directly related to two diagrams one of which is given by the contact graviton-graviton-scalar-scalar vertex (3.27), while the other corresponds to the pole diagram with longitudinal graviton exchange between the Lorentz violating h^3 vertex (3.22) and the ordinary graviton-scalar-scalar vertex (3.26). Again, since ingoing and outgoing gravitons appear transverse ($k_a^\mu \epsilon_\mu(k_a) = 0$, $a = 1, 2$), only the third term in the momentum tensors $P^{\mu\nu}(k_a)$ (3.23) in the h^3 coupling (3.22) contributes to this pole diagram. Apart from that, the most crucial point is that, due to the scalar field energy-momentum tensor conservation, the terms in the inserted graviton propagator (3.21) other than the standard ones (first bracket in (3.21)) give a vanishing result. Keeping all this in mind together with the momenta satisfying $k_1 + k_2 + p_1 + p_2 = 0$ ($k_{1,2}$ and $p_{1,2}$ are the graviton and scalar field 4-momenta, respectively), one readily comes to a simple matrix element for the pole diagram

$$i\mathcal{M}_{pole} = -\frac{2i}{MM_p}\phi(p_2)(\varepsilon_1 \cdot \varepsilon_2)^2 (\mathfrak{n}_{\mu\nu}p_1^\mu p_2^\nu)\phi(p_1) . \tag{3.29}$$

This pole term is precisely cancelled by the contact term, $i\mathcal{M}_{con}$, when the SLIV vertex (3.27) is properly contracted with the graviton polarization vectors and the scalar boson wave functions. Again, we may conclude that physical Lorentz invariance is left intact in graviton scattering on a massive scalar, provided that its dispersion relation is supposed to be recovered when going from the tensor field Lagrangian $\mathcal{L}$ (3.15) to general relativity, as was argued above.

3.2.4 Scalar-scalar scattering

This process, due to graviton exchange, appears in the order $1/M_P^2$ and again is given by an ordinary Lorentz invariant amplitude. As was mentioned above, only the standard terms given by the first bracket in the graviton propagator (3.21) contribute when it is sandwiched between conserved energy-momentum tensors of matter fields. Actually, as one can easily confirm, the contraction of any other term in (3.21) depending on the graviton 4-momentum $k = p_1 + p_2 = -(p_3 + p_4)$ with the graviton-scalar-scalar vertex (3.26) gives a zero result.

3.2.5 Other processes

Many other tree level Lorentz violating processes, related to gravitons and scalar fields (matter fields, in general) appear in higher orders in the basic SLIV parameter $1/M$, by iteration of couplings presented in our basic Lagrangian (3.15) or from a further expansion of the effective Higgs mode (3.14) inserted into the starting Lagrangian (3.2). Again, their amplitudes are essentially determined by an interrelation between the longitudinal graviton exchange diagrams and the corresponding contact multi-graviton interaction diagrams, which appear to cancel each other, thus eliminating physical Lorentz violation in the theory.

Most likely, the same conclusion could be expected for SLIV loop contributions as well. Actually, as in the massless QED case considered earlier (Azatov and Chkareuli, 2006), the corresponding one-loop matrix elements in the Goldstonic gravity theory could either vanish by themselves or amount to the differences between pairs of similar integrals whose integration variables are shifted relative to each other by some constants (being in general arbitrary functions of the external four-momenta of the particles involved) which, in the framework of dimensional regularization, could lead to their total cancelation.

So, the Goldstonic tensor field gravity theory is likely to be physically indistinguishable from conventional general relativity taken in the weak-field limit, provided that the underlying diff invariance is kept exact. This, as we have seen, requires the tensor field gravity to be extended to GR, in order not to otherwise have an actual Lorentz violation in the matter field sector. In this connection, the question arises whether or not the SLIV cancellations continue to work once the tensor field gravity theory is extended to GR, which introduces many additional terms in the starting Lagrangian $\mathcal{L}(H_{\mu\nu}, \phi)$ (3.2). Indeed, since all the new terms are multi-linear in $H_{\mu\nu}$ and contain higher orders in $1/M_P$, the "old" SLIV cancellations (considered above) will not be disturbed, while "new" cancellations will be provided, as one should expect, by an extended diff invariance. This extended diff invariance follows from the proper expansion of the metric transformation law in GR

$$\delta g_{\mu\nu} = \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho} + \xi^\rho \partial_\rho g_{\mu\nu} \tag{3.30}$$

up to the order in which the extended tensor field theory, given by the modified Lagrangian $\mathcal{L}_{ext}(H_{\mu\nu}, \phi)$, is considered.

Chapter 4

Summary and Conclusion

4.1 Result summary for the Yang-Mills theories

The spontaneous Lorentz violation realized through a nonlinear vector field constraint of the type $A^2 = M^2$ (M is the proposed scale for Lorentz violation) is shown to generate massless vector Goldstone bosons gauging the starting global internal symmetries involved, both in the Abelian and the non-Abelian symmetry case. The gauge invariance, as we have seen, directly follows from a general variation principle and Noether's second theorem (Noether, 1918), as a necessary condition for these bosons not to be superfluously restricted in degrees of freedom once the true vacuum in a theory is chosen by the SLIV constraint. It should be stressed that we can of course only achieve this derivation of gauge invariance by allowing all the coupling constants in the Lagrangian density to be determined from the requirement of avoiding any extra restriction imposed on the vector field(s) in addition to the SLIV constraint. Actually, this derivation excludes "wrong" couplings in the vector field Lagrangian, which would otherwise distort the final Lorentz symmetry broken phase with unphysical extra states including ghost-like ones. Note that this procedure might, in some sense, be inspired by string theory where the coupling constants are just vacuum expectation values of the dilaton and moduli fields[1]. So, the adjustment of coupling constants in the Lagrangian would mean, in essence, a certain choice for the vacuum configurations of these fields, which are thus correlated with the SLIV. Another important point for this gauge symmetry derivation is that we followed our philosophy of imposing the SLIV constraints, (2.2) and (2.14) respectively, without adding a Lagrange multiplier term, as one might have imagined should come with these constraints. Had we done so the equations of motion would have changed and the Lagrange multiplier might have picked up the inconsistency, which we required to be solved in the Abelian case by Eq. (2.4) and in the non-Abelian case by Eq. (2.17).

In the Abelian case a massless vector Goldstone boson appears, which is naturally associated with the photon. In the non-Abelian case it was shown that the pure Lorentz violation still generates just one genuine Goldstone vector boson. However the SLIV constraint (2.14) manifests a larger accidental $SO(D, 3D)$ symmetry, which is not shared by the Lagrangian $\mathcal{L}$. The spontaneous violation of this $SO(D, 3D)$ symmetry generates $D - 1$ pseudo-Goldstone vector bosons which, together with the genuine Goldstone vector boson, complete the whole gauge field multiplet of the internal symmetry group G. Remarkably, these vector bosons all appear to be strictly massless, as they are protected by the simultaneously generated non-Abelian gauge invariance. These theories, both Abelian and non-Abelian, though being essentially nonlinear, appear to be physically indistinguishable from the conventional QED and Yang-Mills theories due to their generic, SLIV enforced, gauge invariance. One could actually see that just this gauge invariance ensures that our theories do not have unreasonably large (proportional to the SLIV scale M) Lorentz violation in the fermion and vector field interaction terms. It appears also to ensure that all the physical Lorentz violating effects,

[1]Green et al. (b,a)

even those suppressed by this SLIV scale, are non-observable.

In this connection, the only way for physical Lorentz violation then to appear would be if the above gauge invariance is somehow broken at very small distances. One could imagine how such a breaking might occur. Only gauge invariant theories provide, as we have learned, the needed number of degrees of freedom for the interacting vector fields once the SLIV occurs. Note that a superfluous restriction on a vector (or any other) field would make it impossible to set the required initial conditions in the appropriate Cauchy problem and, in quantum theory, to choose self-consistent equal-time commutation relations[2]. One could expect, however, that gravity could in general hinder the setting of the required initial conditions at extra-small distances. Eventually this would manifest itself in the violation of the above gauge invariance in a theory through some high-order operators stemming from the gravity-influenced area, which could lead to physical Lorentz violation. We may return to this interesting possibility elsewhere.

[2]Ogievetsky and Polubarinov (1965)

4.2 Result Summary for the Tensor Field Gravity

The spontaneous Lorentz violation have been considered, appearing through the length-fixing tensor field constraint $H_{\mu\nu}^2 = \pm M^2$ (M is the proposed scale for Lorentz violation), in the tensor field gravity theory which mimics general relativity in Minkowski space-time. We have shown that such a SLIV pattern, due to which the true vacuum in the theory is chosen, induces massless tensor Goldstone modes some of which can naturally be associated with the physical graviton. This theory looks essentially nonlinear and contains a variety of Lorentz and CPT violating couplings, when expressed in terms of the pure tensor Goldstone modes. Nonetheless, all the SLIV effects turn out to be strictly cancelled in the lowest order graviton-graviton scattering, due to the diff invariance of the free tensor field Lagrangian (3.3) we started with. At the same time, actual Lorentz violation may appear in the matter field interaction sector (3.5), which only possesses an approximate diff invariance, through deformed dispersion relations of the matter fields involved. However, a proper extension of the tensor field theory to GR, with its exact diff invariance, ultimately restores the normal dispersion relations for matter fields and, therefore, the SLIV effects vanish. So, as we generally argue, the measurable effects of SLIV, induced by elementary vector or tensor fields, can be related to the accompanying gauge symmetry breaking rather than to spontaneous Lorentz violation. The latter appears by itself to be physically unobservable and only results in a non-covariant gauge choice in an otherwise gauge invariant and Lorentz invariant theory.

From this standpoint, the only way for physical Lorentz violation to appear would be if the above local invariance is slightly broken at very small distances. This is in fact a place where the Goldstonic vector and tensor field theories drastically differ from conventional QED, Yang-Mills and GR theories. Actually, such a local symmetry breaking could lead in the former case to deformed dispersion relations for all the matter fields involved. This effect typically appears proportional to some power of the ratio $\frac{M}{M_P}$ (just as we have seen above for the scalar field in our model, see (3.17)), though being properly suppressed by tiny gauge non-invariance. Remarkably, the higher the SLIV scale M becomes the larger becomes the actual Lorentz violation which, for some value of the scale M, may become physically observable even at low energies. Another basic distinction of Goldstonic theories with non-exact gauge invariance is the emergence of a mass for the graviton and other gauge fields (namely, for the non-Abelian ones, see[3]), if they are composed from Pseudo Goldstone modes rather than from pure Goldstone ones. Indeed, these PGMs are no longer protected by gauge invariance and may properly acquire tiny masses, which still do not contradict experiment. This may lead to a massive gravity theory where the graviton mass emerges dynamically, thus avoiding the notorious discontinuity problem[4]. So, while Goldstonic theories with exact local invariance are physically indistinguishable from conventional gauge theories, there are some principal distinctions when this local symmetry is slightly broken which could eventually allow us to differentiate between the two types of theory in an observational way.

One could imagine how such a local symmetry breaking might occur. As was earlier argued[5], only local invariant theories provide the needed number of degrees of freedom for interacting gauge fields once SLIV occurs. Note that a superfluous restriction put on vector or tensor fields would make it impossible to set the required initial conditions in the appropriate Cauchy problem and, in quantum theory, to choose self-consistent equal-time commutation relations[6]. One could expect, however, that quantum gravity could in general hinder the setting of the required initial conditions at extra-small distances. Eventually, this would manifest itself in violation of the above local invariance in a theory through some high-order operators stemming from the quantum gravity influenced area, which could lead to physical Lorentz violation. This attractive point seems to deserve further consideration.

[3] Chkareuli and Jejelava (2008)

[4] van Dam and Veltman (1970)

[5] Chkareuli et al. (2008)

[6] Ogievetsky and Polubarinov (1965)

Chapter 5

Appendix A

To calculating the graviton propagator it was necessary to gather more than thirteen hundred terms of the different type. In theory in case of motivation and lots of free time it was possible to perform all calculations by hand, However it is also possible to choose faster and trustable way. One could use particular gear, the special computing machine which is commonly known as Personal Computer.

Nevertheless, considering current case , we had to make sure of all calculations twice (both by hand and by computing) For verification reasons of the code correctness's.

This script listed below had been written in a way providing the ability to manipulate 4-tesor and 4-vectors. Current version of script contains the part for complicate tensor expression analyzes. We are reporting the source code *(originally written for the Wolfram Mathematica 7.0)* directly extracted from nb file format to tex.

```
*)

MetricTensorRule :={  (*
```

Here is the rule for the contracting the metric tensor with itself, vectors and tensors(with two indexes).

(Although all used tensors are symmetric, it is easy to modify main code to generalize the ability of the program). .*)

(*$g_{\mu\rho}g^{\rho\nu}$*)

(*$g_{\mu\rho}g^{\nu\rho} \to g_\mu{}^\nu$*)MTD[$\mu$_, ρ_]MTU[ν_, ρ_]->MTM[μ, ν],

(*$g_{\mu\rho}g^{\rho\nu} \to g_\mu{}^\nu$*)MTD[$\mu$_, ρ_]MTU[ρ_, ν_]->MTM[μ, ν],

(*$g_{\rho\mu}g^{\rho\nu} \to g_\mu{}^\nu$*)MTD[$\rho$_, μ_]MTU[ρ_, ν_]->MTM[μ, ν],

(*$g_{\rho\mu}g^{\nu\rho} \to g_\mu{}^\nu$*)MTD[$\rho$_, μ_]MTU[ν_, ρ_]->MTM[μ, ν],

(*$g_{\mu\nu}$*)

(*$g_{\mu\rho}k^\rho \to k_\mu$*)MTD[$\mu$_, ρ_]VU[k_, ρ_]->VD[k, μ],

(*$g_{\rho\mu}k^\rho \to k_\mu$*)MTD[$\rho$_, μ_]VU[k_, ρ_]->VD[k, μ],

(*$g_{\mu\rho}\delta_\nu{}^\rho \to g_{\mu\nu}$*)MTD[$\mu$_, ρ_]MTM[ν_, ρ_]->MTD[μ, ν],

(*$g_{\rho\mu}\delta_\nu{}^\rho \to g_{\mu\nu}$*)MTD[$\rho$_, μ_]MTM[ν_, ρ_]->MTD[μ, ν],

(*$g_{\mu\rho}n^{\nu\rho} \to n_\mu{}^\nu$*)MTD[$\mu$_, ρ_]TU[n_, ν_, ρ_]->TM[n, μ, ν],

(*$g_{\rho\mu}n^{\nu\rho} \to n_\mu{}^\nu$*)MTD[$\rho$_, μ_]TU[n_, ν_, ρ_]->TM[n, μ, ν],

(*$g_{\mu\rho}n^{\rho\nu} \to n_\mu{}^\nu$*)MTD[$\mu$_, ρ_]TU[n_, ρ_, ν_]->TM[n, μ, ν],

(*$g_{\rho\mu}n^{\rho\nu} \to n_\mu{}^\nu$*)MTD[$\rho$_, μ_]TU[n_, ρ_, ν_]->TM[n, μ, ν],

$(*g_{\mu\rho}n_\nu{}^\rho \to n_{\mu\nu}*)$MTD[$\mu$_, ρ_]TM[n_, ν_, ρ_]->TD[n, μ, ν],

$(*g_{\rho\mu}n_\nu{}^\rho \to n_{\mu\nu}*)$MTD[$\rho$_, μ_]TM[n_, ν_, ρ_]->TD[n, μ, ν],

$(*g^{\mu\nu}*)$

$(*g^{\mu\rho}k_\rho \to k^\mu*)$MTU[$\mu$_, ρ_]VD[k_, ρ_]->VU[k, μ],

$(*g^{\rho\mu}k_\rho \to k^\mu*)$MTU[$\rho$_, μ_]VD[k_, ρ_]->VU[k, μ],

$(*g^{\mu\rho}\delta_\rho{}^\nu \to g^{\mu\nu}*)$MTU[$\mu$_, ρ_]MTM[ρ_, ν_]->MTU[μ, ν],

$(*g^{\rho\mu}\delta_\rho{}^\nu \to g^{\mu\nu}*)$MTU[$\rho$_, μ_]MTM[ρ_, μ_]->MTU[μ, ν],

$(*g^{\mu\rho}n_{\nu\rho} \to n_\nu{}^\mu*)$MTU[$\mu$_, ρ_]TD[n_, ν_, ρ_]->TM[n, ν, μ],

$(*g^{\rho\mu}n_{\nu\rho} \to n_\nu{}^\mu*)$MTU[$\rho$_, μ_]TD[n_, ν_, ρ_]->TM[n, ν, μ],

$(*g^{\mu\rho}n_{\rho\nu} \to n_\nu{}^\mu*)$MTU[$\mu$_, ρ_]TD[n_, ρ_, ν_]->TM[n, ν, μ],

$(*g^{\rho\mu}n_{\rho\nu} \to n_\nu{}^\mu*)$MTU[$\rho$_, μ_]TD[n_, ρ_, ν_]->TM[n, ν, μ],

$(*g^{\mu\rho}n_\rho{}^\nu \to n^{\mu\nu}*)$MTU[$\mu$_, ρ_]TM[n_, ρ_, ν_]->TU[n, μ, ν],

$(*g^{\rho\mu}n_\rho{}^\nu \to n^{\mu\nu}*)$MTU[$\rho$_, μ_]TM[n_, ρ_, ν_]->TU[n, μ, ν],

$(*\delta_\mu{}^\nu*)$

$(*\delta_\rho{}^\mu k^\rho \to k^\mu*)$MTM[$\rho$_, μ_]VU[k_, ρ_]->VU[k, μ],

(*$\delta_\mu{}^\rho k_\rho \to k_\mu$*)MTM[$\mu$_, ρ_]VD[k_, ρ_]->VD[k, μ],

(*$\delta_\mu{}^\rho g_{\rho\nu} \to g_{\mu\nu}$*)MTM[$\mu$_, ρ_]MTD[ρ_, ν_]->MTD[μ, ν],

(*$\delta_\mu{}^\rho g_{\nu\rho} \to g_{\nu\mu}$*)MTM[$\mu$_, ρ_]MTD[ν_, ρ_]->MTD[ν, μ],

(*$\delta_\rho{}^\mu g^{\rho\nu} \to g^{\mu\nu}$*)MTM[$\rho$_, μ_]MTU[ρ_, ν_]->MTU[μ, ν],

(*$\delta_\rho{}^\mu g^{\nu\rho} \to g^{\nu\mu}$*)MTM[$\rho$_, μ_]MTU[ν_, ρ_]->MTU[ν, μ],

(*$\delta_\mu{}^\rho \delta_\rho{}^\nu \to \delta_\mu{}^\nu$*)MTM[$\mu$_, ρ_]MTM[ρ_, ν_] $\to$ MTM[μ, ν],

(*$\delta_\nu{}^\rho \delta_\rho{}^\mu \to \delta_\nu{}^\mu$*)MTM[$\nu$_, ρ_]MTM[ρ_, μ_] $\to$ MTM[ν, μ],

(*$\delta_\mu{}^\rho n_{\rho\nu} \to n_{\mu\nu}$*)MTM[$\mu$_, ρ_]TD[n_, ρ_, ν_] $\to$ TD[n, μ, ν],

(*$\delta_\mu{}^\rho n_{\nu\rho} \to n_{\nu\mu}$*)MTM[$\mu$_, ρ_]TD[n_, ν_, ρ_] $\to$ TD[n, ν, μ],

(*$\delta_\rho{}^\mu n^{\rho\nu} \to n^{\mu\nu}$*)MTM[$\rho$_, μ_]TU[n_, ρ_, ν_] $\to$ TU[n, μ, ν],

(*$\delta_\rho{}^\mu n^{\nu\rho} \to n^{\nu\mu}$*)MTM[$\rho$_, μ_]TU[n_, ν_, ρ_] $\to$ TU[n, ν, μ],

(*$\delta_\rho{}^\mu n_\nu{}^\rho \to n_\nu{}^\mu$*)MTM[$\rho$_, μ_]TM[n_, ν_, ρ_] $\to$ TM[n, ν, μ],

(*$\delta_\mu{}^\rho n_\rho{}^\nu \to n_\mu{}^\nu$*)MTM[$\mu$_, ρ_]TM[n_, ρ_, ν_] $\to$ TM[n, μ, ν],

(*$n_\rho{}^\tau n_\tau{}^\rho \to$< nn > *)TM[n_, τ_, ρ_]TM[n_, ρ_, τ_]->AngleBracket[n, n], (*Pleasechecktwice!!!*)

(*$n_{\rho\tau} n^{\tau\rho} \to$< nn > *)TD[n_, ρ_, τ_]TU[n_, τ_, ρ_]->AngleBracket[n, n], (*Pleasechecktwice!!!*)

(*$n_{\rho\tau}n^{\rho\tau} \to <$ nn $>$ *)TD[n_, ρ_, τ_]TU[n_, ρ_, τ_]->AngleBracket[n, n], (*Pleasechecktwice!!!*)

(*$n_{\mu\rho}n^{\nu\rho} \to < n_\mu n^\nu >$ *)TD[n_, μ_, ρ_]TU[n_, ν_, ρ_]->AngleBracket[Subscript[n, μ]Superscript[n, ν]],

(*$n_{\mu\rho}n^{\rho\nu} \to < n_\mu n^\nu >$ *)TD[n_, μ_, ρ_]TU[n_, ρ_, ν_]->AngleBracket[Subscript[n, μ]Superscript[n, ν]],

(*$n_{\rho\mu}n^{\nu\rho} \to < n_\mu n^\nu >$ *)TD[n_, ρ_, μ_]TU[n_, ν_, ρ_]->AngleBracket[Subscript[n, μ]Superscript[n, ν]],

(*$n_{\rho\mu}n^{\rho\nu} \to < n_\mu n^\nu >$ *)TD[n_, ρ_, μ_]TU[n_, ρ_, ν_]->AngleBracket[Subscript[n, μ]Superscript[n, ν]],

(*${n_\mu}^\rho {n_\rho}^\nu \to < n_\mu n^\nu >$ *)TM[n_, μ_, ρ_]TM[n_, ρ_, ν_]->AngleBracket[Subscript[n, μ]Superscript[n, ν]],

(*$n_{\mu\rho}{n_\nu}^\rho \to < n_\mu n_\nu >$ *)TD[n_, μ_, ρ_]TM[n_, ν_, ρ_]->AngleBracket[Subscript[n, μ]Subscript[n, ν]],

(*$n_{\rho\mu}{n_\nu}^\rho \to < n_\mu n_\nu >$ *)TD[n_, ρ_, μ_]TM[n_, ν_, ρ_]->AngleBracket[Subscript[n, μ]Subscript[n, ν]],

(*$n^{\mu\rho}{n_\rho}^\nu \to < n^\mu n^\nu >$ *)TU[n_, μ_, ρ_]TM[n_, ρ_, ν_]->AngleBracket[Superscript[n, μ]Superscript[n, ν]],

(*$n^{\rho\mu}{n_\rho}^\nu \to < n^\mu n^\nu >$ *)TU[n_, ρ_, μ_]TM[n_, ρ_, ν_]->AngleBracket[Superscript[n, μ]Superscript[n, ν]]

}

FullScalarProductRule:={(*Contract Tensor to itself*)

(*${\delta_\rho}^\rho \to 4$*)MTM[ρ_, ρ_] $\to$ 4,

(*$k_\rho k^\rho \to k^2$*)VD[k_, ρ_]VU[k_, ρ_] $\to k^2$,

(*$q_\rho k^\rho \to <$ qk $>$ *)VD[q_, ρ_]VU[k_, ρ_] $\to$ AngleBracket[qk],

(*$k_\rho q^\rho \to <$ kq $>$ *)VD[k_, ρ_]VU[k_, ρ_] $\to$ AngleBracket[qk],

(*${n_\rho}^\rho \to < n >$ *)TM[n_, ρ_, ρ_] $\to$ AngleBracket[n],

(* $< n > \to 0$*)(*AngleBracket[n] $\to$ 0,*)

(*$n_{\tau\rho}q^{\rho}k^{\tau}$ $\to <$ nqk $>$ *)TD[n_, τ_, ρ_]VU[q_, ρ_]VU[k_, τ_]->AngleBracket[n, qk],

(*$n_{\tau\rho}q^{\tau}k^{\rho}$ $\to <$ nqk $>$ *)TD[n_, τ_, ρ_]VU[q_, τ_]VU[k_, ρ_]->AngleBracket[n, qk],

(*$n^{\tau\rho}q_{\rho}k_{\tau}$ $\to <$ nqk $>$ *)TU[n_, τ_, ρ_]VD[q_, ρ_]VD[k_, τ_]->AngleBracket[n, qk],

(*$n^{\tau\rho}q_{\tau}k_{\rho}$ $\to <$ nqk $>$ *)TU[n_, τ_, ρ_]VD[q_, τ_]VD[k_, ρ_]->AngleBracket[n, qk],

(*$n_{\tau}{}^{\rho}q_{\rho}k^{\tau}$ $\to <$ nqk $>$ *)TD[n_, τ_, ρ_]VD[q_, ρ_]VU[k_, τ_]->AngleBracket[n, qk],

(*$n_{\tau}{}^{\rho}q_{\rho}k^{\tau}$ $\to <$ nqk $>$ *)TD[n_, τ_, ρ_]VD[q_, ρ_]VU[k_, τ_]->AngleBracket[n, qk]

}

PartialScalarProductRule:={

(*$n_{\mu\rho}k^{\rho} \to < n_{\mu}, k >$ *)TD[n_, μ_, ρ_]VU[k_, ρ_]->AngleBracket[VD[n, μ], k],

(*$n_{\rho\mu}k^{\rho} \to < n_{\mu}, k >$ *)TD[n_, ρ_, μ_]VU[k_, ρ_]->AngleBracket[VD[n, μ], k],

(*$n^{\mu\rho}k_{\rho} \to < n^{\mu}, k >$ *)TU[n_, μ_, ρ_]VD[k_, ρ_]->AngleBracket[VU[n, μ], k],

(*$n^{\rho\mu}k_{\rho} \to < n^{\mu}, k >$ *)TU[n_, ρ_, μ_]VD[k_, ρ_]->AngleBracket[VU[n, μ], k],

(*$n_{\mu}{}^{\rho}k_{\rho} \to < n_{\mu}, k >$ *)TM[n_, μ_, ρ_]VD[k_, ρ_]->AngleBracket[VD[n, μ], k],

(*$n_{\rho}{}^{\mu}k^{\rho} \to < n^{\mu}, k >$ *)TM[n_, ρ_, μ_]VU[k_, ρ_]->AngleBracket[VU[n, μ], k],

(* $< n^{\rho}, q >< m_{\rho}, k > \to < q, n, m, k >$ *)

AngleBracket[VU[n_, ρ_], q_]AngleBracket[VD[m_, ρ_], k_]->AngleBracket[q, n, m, k],

(* $< n_\rho, q >< m^\rho, k >\rightarrow< q, n, m, k >$ *)

AngleBracket[VD[n_, ρ_], q_]AngleBracket[VU[m_, ρ_], k_]->AngleBracket[q, n, m, k],

(* Complicate scalar production *)

(*begin*)

(* $< n^\mu n^\rho > k_\rho \rightarrow< n^\mu, n, k >$ *)

AngleBracket[Superscript[n_, μ_]Superscript[n_, ρ_]]VD[k_, ρ_] → AngleBracket[VU[n, μ], n, k],

(* $< n_\mu n^\rho > k_\rho \rightarrow< n_\mu, n, k >$ *)

AngleBracket[Subscript[n_, μ_]Superscript[n_, ρ_]]VD[k_, ρ_] → AngleBracket[VD[n, μ], n, k],

(* $< n_\mu n_\rho > k^\rho \rightarrow< n^\mu, n, k >$ *)

AngleBracket[Subscript[n_, μ_]Subscript[n_, ρ_]]VU[k_, ρ_] → AngleBracket[VD[n, μ], n, k],

(* $< n^\mu n_\rho > k^\rho \rightarrow< n^\mu, n, k >$ *)

AngleBracket[Superscript[n_, μ_]Subscript[n_, ρ_]]VU[k_, ρ_] → AngleBracket[VU[n, μ], n, k],

(*$m^{\mu\rho} < n_\rho k >\rightarrow< m^\mu, n, k >$ *)

TU[m_, μ_, ρ_]AngleBracket[VD[n_, ρ_], k_]->AngleBracket[VU[m, μ], n, k],

(*$m^{\rho\mu} < n_\rho k >\rightarrow< m^\mu, n, k >$ *)

TU[m_, ρ_, μ_]AngleBracket[VD[n_, ρ_], k_]->AngleBracket[VU[m, μ], n, k],

(*$m_{\mu\rho} < n^{\rho}k > \to < m_{\mu}, n, k >$ *)

TD[m_, μ_, ρ_]AngleBracket[VU[n_, ρ_], k_]->AngleBracket[VD[m, μ], n, k],

(*$m_{\rho\mu} < n^{\rho}k > \to < m_{\mu}, n, k >$ *)

TD[m_, ρ_, μ_]AngleBracket[VU[n_, ρ_], k_]->AngleBracket[VD[m, μ], n, k],

(*$m_{\mu}{}^{\rho} < n_{\rho}k > \to < m_{\mu}, n, k >$ *)

TM[m_, μ_, ρ_]AngleBracket[VD[n_, ρ_], k_]->AngleBracket[VD[m, μ], n, k],

(*$m_{\rho}{}^{\mu} < n^{\rho}k > \to < m^{\mu}, n, k >$ *)

TM[m_, ρ_, μ_]AngleBracket[VU[n_, ρ_], k_]->AngleBracket[VU[m, μ], n, k],

(* $< m^{\rho}, n, k > q_{\rho} \to < q, m, n, k >$ *)

AngleBracket[VU[m_, ρ_], n_, k_]VD[q_, ρ_] → AngleBracket[q, m, n, k],

(* $< m_{\rho}, n, k > q^{\rho} \to < q, m, n, k >$ *)

AngleBracket[VD[m_, ρ_], n_, k_]VU[q_, ρ_] → AngleBracket[q, m, n, k]

(*end*)

}

ComplicateScalarProductRule:={

(* $< m^{\rho}, n, k >< l_{\rho}, q > \to < q, l, m, n, k >$ *)

AngleBracket[VU[m_, ρ_], n_, k_]AngleBracket[VD[l_, ρ_], q_] → AngleBracket[q, l, m, n, k],

(* $< m_{\rho}, n, k >< l^{\rho}, q > \to < q, l, m, n, k >$ *)

AngleBracket[VD[m_, ρ_], n_, k_]AngleBracket[VU[l_, ρ_], q_] → AngleBracket[q, l, m, n, k],

(* $< m^{\rho}, t, n, k >< l_{\rho}, k > \to k^2 \;\; < q, l, m, t, n, k >$ *)

AngleBracket[VU[m_, ρ_], t_, n_, k_]AngleBracket[VD[l_, ρ_], k_] → k^2,

(*

$< m_\rho, t, n, k >< l^\rho, k > \to k^2 \quad < q, l, m, t, n, k >$ *)

AngleBracket[VD[m_, ρ_], t_, n_, k_]AngleBracket[VU[l_, ρ_], k_] → k^2,

(* $< n^\mu n^\rho >< m_\rho q > \to < n^\mu, n, m, k >$ *)

AngleBracket[Superscript[n_, μ_]Superscript[n_, ρ_]]AngleBracket[VD[m_, ρ_], q_] →

AngleBracket[VU[n, μ], n, m, q],

(* $< n_\mu n^\rho >< m_\rho q > \to < n^\mu, n, m, k >$ *)

AngleBracket[Subscript[n_, μ_]Superscript[n_, ρ_]]AngleBracket[VD[m_, ρ_], q_] →

AngleBracket[VD[n, μ], n, m, q],

(* $< n_\mu n_\rho >< m^\rho q > \to < n^\mu, n, m, k >$ *)

AngleBracket[Subscript[n_, μ_]Subscript[n_, ρ_]]AngleBracket[VU[m_, ρ_], q_] →

AngleBracket[VD[n, μ], n, m, q],

(* $< n^\mu n_\rho >< m^\rho q > \to < n^\mu, n, m, k >$ *)

AngleBracket[Superscript[n_, μ_]Subscript[n_, ρ_]]AngleBracket[VU[m_, ρ_], q_] →

AngleBracket[VU[n, μ], n, m, q],

(* $< n^\rho, n, n_\rho > \to < n, n, n >$ *)AngleBracket[VU[n_, ρ_], n_, VD[m_, ρ_]]->AngleBracket[n, n, n],

(* $< n_\rho, n, n^\rho > \to < n, n, n >$ *)AngleBracket[VD[n_, ρ_], n_, VU[m_, ρ_]]->AngleBracket[n, n, n],

(* $< n^\mu n^\rho > m_{\rho\nu} \to < n^\mu, n, m_\nu >$ *)

AngleBracket[Superscript[n_, μ_]Superscript[n_, ρ_]]TD[m_, ρ_, ν_]->

AngleBracket[VU[n, μ], n, VD[m, ν]],

(* $< n^{\mu} n^{\rho} > m_{\nu\rho} \rightarrow < n^{\mu}, n, m_{\nu} >$ *)

AngleBracket[Superscript[n_, μ_]Superscript[n_, ρ_]]TD[m_, ν_, ρ_]->

AngleBracket[VU[n, μ], n, VD[m, ν]],

(* $< n^{\mu} n^{\rho} > m_{\rho}{}^{\nu} \rightarrow < n^{\mu}, n, m^{\nu} >$ *)

AngleBracket[Superscript[n_, μ_]Superscript[n_, ρ_]]TM[m_, ρ_, ν_]->

AngleBracket[VU[n, μ], n, VU[m, ν]],

(* $< n_{\mu} n_{\rho} > m^{\rho\nu} \rightarrow < n_{\mu}, n, m^{\nu} >$ *)

AngleBracket[Subscript[n_, μ_]Subscript[n_, ρ_]]TU[m_, ρ_, ν_]->

AngleBracket[VD[n, μ], n, VU[m, ν]],

(* $< n_{\mu} n_{\rho} > m^{\nu\rho} \rightarrow < n^{\mu}, n, m^{\nu} >$ *)

AngleBracket[Subscript[n_, μ_]Subscript[n_, ρ_]]TU[m_, ν_, ρ_]->

AngleBracket[VD[n, μ], n, VU[m, ν]],

(* $< n_{\mu} n_{\rho} > m_{\nu}{}^{\rho} \rightarrow < n_{\mu}, n, m_{\nu} >$ *)

AngleBracket[Subscript[n_, μ_]Subscript[n_, ρ_]]TM[m_, ν_, ρ_]->

AngleBracket[VD[n, μ], n, VD[m, ν]],

(* $< n^{\mu} n_{\rho} > m^{\nu\rho} \rightarrow < n^{\mu}, n, m_{\nu} >$ *)

AngleBracket[Superscript[n_, μ_]Subscript[n_, ρ_]]TU[m_, ν_, ρ_]->

AngleBracket[VU[n, μ], n, VU[m, ν]],

(* $< n^{\mu} n_{\rho} > m^{\nu\rho} \rightarrow < n^{\mu}, n, m_{\nu} >$ *)

AngleBracket[Superscript[n_, μ_]Subscript[n_, ρ_]]TU[m_, ρ_, ν_]->

AngleBracket[VU[n, μ], n, VU[m, ν]],

$(* < n_\mu n^\rho > m_{\rho\nu} \to < n_\mu, n, m^\nu > *)$

AngleBracket[Subscript[n_, μ_]Superscript[n_, ρ_]]TD[m_, ρ_, ν_]->

AngleBracket[VD[n, μ], n, VD[m, ν]],

$(* < n_\mu n^\rho > m_{\rho\nu} \to < n_\mu, n, m^\nu > *)$

AngleBracket[Subscript[n_, μ_]Superscript[n_, ρ_]]TD[m_, ν_, ρ_]->

AngleBracket[VD[n, μ], n, VD[m, ν]],

(*

$< n_\mu n^\rho > m_\rho{}^\nu \to < n_\mu, n, m^\nu > *)$AngleBracket[Subscript[n_, μ_]Superscript[n_, ρ_]]TM[m_, ρ_, ν_]->

AngleBracket[VD[n, μ], n, VU[m, ν]],

$(* < n^\mu n_\rho > m_\nu{}^\rho \to < n^\mu, n, m_\nu > *)$

AngleBracket[Superscript[n_, μ_]Subscript[n_, ρ_]]TM[m_, ν_, ρ_]->

AngleBracket[VU[n, μ], n, VD[m, ν]],

$(* m^{\mu\rho} < n_\rho, n, k > \to < m^\mu, n, n, k > *)$

TU[m_, μ_, ρ_]AngleBracket[VD[n_, ρ_], n_, k_]->AngleBracket[VU[m, μ], n, n, k],

$(* m^{\rho\mu} < n_\rho, n, k > \to < m^\mu, n, n, k > *)$

TU[m_, ρ_, μ_]AngleBracket[VD[n_, ρ_], n_, k_]->AngleBracket[VU[m, μ], n, n, k],

$(* m_{\mu\rho} < n^\rho, n, k > \to < m_\mu, n, n, k > *)$

TD[m_, μ_, ρ_]AngleBracket[VU[n_, ρ_], n_, k_]->AngleBracket[VD[m, μ], n, n, k],

$(* m_{\rho\mu} < n^\rho, n, k > \to < m_\mu, n, n, k > *)$

TD[m_, ρ_, μ_]AngleBracket[VU[n_, ρ_], n_, k_]->AngleBracket[VD[m, μ], n, n, k],

$(*m_\mu{}^\rho < n_\rho, n, k > \to < m_\mu, n, n, k > *)$

TM[m_, μ_, ρ_]AngleBracket[VD[n_, ρ_], n_, k_]->AngleBracket[VD[m, μ], n, n, k],

$(*m_\rho{}^\mu < n^\rho, n, k > \to < m^\mu, n, n, k > *)$

TM[m_, ρ_, μ_]AngleBracket[VU[n_, ρ_], n_, k_]->AngleBracket[VU[m, μ], n, n, k],

$(*m^{\mu\rho} < n_\rho, n, n, k > \to k^\mu \; < m^\mu, n, n, n, k > *)$

TU[m_, μ_, ρ_]AngleBracket[VD[n_, ρ_], n_, n_, k_]->VU[k, μ],

$(*m^{\rho\mu} < n_\rho, n, n, k > \to k^\mu \; < m^\mu, n, n, n, k > *)$

TU[m_, ρ_, μ_]AngleBracket[VD[n_, ρ_], n_, n_, k_]->VU[k, μ],

$(*m_{\mu\rho} < n^\rho, n, n, k > \to k_\mu \; < m_\mu, n, n, n, k > *)$

TD[m_, μ_, ρ_]AngleBracket[VU[n_, ρ_], n_, n_, k_]->VD[k, μ],

$(*m_{\rho\mu} < n^\rho, n, n, k > \to k_\mu \; < m_\mu, n, n, n, k > *)$

TD[m_, ρ_, μ_]AngleBracket[VU[n_, ρ_], n_, n_, k_]->VD[k, μ],

$(*m_\mu{}^\rho < n_\rho, n, n, k > \to k_\mu \; < m_\mu, n, n, n, k > *)$

TM[m_, μ_, ρ_]AngleBracket[VD[n_, ρ_], n_, n_, k_]->VD[k, μ],

$(*m_\rho{}^\mu < n^\rho, n, n, k > \to k^\mu \; < m^\mu, n, n, n, k > *)$

TM[m_, ρ_, μ_]AngleBracket[VU[n_, ρ_], n_, n_, k_]->VU[k, μ],

$(* < n^{\mu} n^{\rho} >< n_{\rho}, k > \to < n^{\mu}, n, n, k > *)$

$\text{AngleBracket}[\text{Superscript}[\text{n}\_, \mu\_]\text{Superscript}[\text{n}\_, \rho\_]]\text{AngleBracket}[\text{VD}[\text{n}\_, \rho\_], k]$

$\to \text{AngleBracket}[\text{VD}[n, \mu], n, n, k],$

$(* < n_{\mu} n^{\rho} >< n_{\rho}, k > \to < n_{\mu}, n, n, k > *)$

$\text{AngleBracket}[\text{Subscript}[\text{n}\_, \mu\_]\text{Superscript}[\text{n}\_, \rho\_]]\text{AngleBracket}[\text{VD}[\text{n}\_, \rho\_], k]$

$\to \text{AngleBracket}[\text{VD}[n, \mu], n, n, k],$

$(* < n_{\mu} n_{\rho} >< n^{\rho}, k > \to < n^{\mu}, n, n, k > *)$

$\text{AngleBracket}[\text{Subscript}[\text{n}\_, \mu\_]\text{Subscript}[\text{n}\_, \rho\_]]\text{AngleBracket}[\text{VU}[\text{n}\_, \rho\_], k]$

$\to \text{AngleBracket}[\text{VU}[n, \mu], n, n, k],$

$(* < n^{\mu} n_{\rho} >< n^{\rho}, k > \to < n^{\mu}, n, n, k > *)$

$\text{AngleBracket}[\text{Superscript}[\text{n}\_, \mu\_]\text{Subscript}[\text{n}\_, \rho\_]]\text{AngleBracket}[\text{VU}[\text{n}\_, \rho\_], k]$

$\to \text{AngleBracket}[\text{VU}[n, \mu], n, n, k],$

$(* < n^{\mu} n^{\rho} >< n_{\rho}, n, k > \to k^{\mu} \ < n^{\mu}, n, n, n, k > *)$

$\text{AngleBracket}[\text{Superscript}[\text{n}\_, \mu\_]\text{Superscript}[\text{n}\_, \rho\_]]\text{AngleBracket}[\text{VD}[\text{n}\_, \rho\_], n, k] \to \text{VU}[k, \mu],$

$(* < n_{\mu} n^{\rho} >< n_{\rho}, n, k > \to k_{\mu} \ < n_{\mu}, n, n, n, k > *)$

$\text{AngleBracket}[\text{Subscript}[\text{n}\_, \mu\_]\text{Superscript}[\text{n}\_, \rho\_]]\text{AngleBracket}[\text{VD}[\text{n}\_, \rho\_], n, k] \to \text{VD}[k, \mu],$

$(* < n_{\mu} n_{\rho} >< n^{\rho}, n, k > \to k_{\mu} \ < n^{\mu}, n, n, n, k > *)$

$\text{AngleBracket}[\text{Subscript}[\text{n}\_, \mu\_]\text{Subscript}[\text{n}\_, \rho\_]]\text{AngleBracket}[\text{VU}[\text{n}\_, \rho\_], n, k] \to \text{VU}[k, \mu],$

$(* < n^{\mu} n_{\rho} >< n^{\rho}, n, k > \to k^{\mu} \ < n^{\mu}, n, n, n, k > *)$

$\text{AngleBracket}[\text{Superscript}[\text{n}\_, \mu\_]\text{Subscript}[\text{n}\_, \rho\_]]\text{AngleBracket}[\text{VU}[\text{n}\_, \rho\_], n, k] \to \text{VU}[k, \mu]$

}

```
(* This part of code identify tensor structure *)

IdentifyTensor[z_]:=Module[{iiii},


TensorStruct = 1;

(*find structure with mask < n_μ, ... > or < ..., n_μ > or < n^μ, ... > or < ..., n^μ > or couple of them*)

For[iiii = 1, iiii<=Length[z], iiii++,

{

If[(* try to check the appearance of the tensor structure in Angle Bracket *)

Head[Part[z, iiii]] == AngleBracket,

{If[(* Identify Tensor Structure (TS) to the beginning of bracket *)

Or[Head[Part[z, iiii, 1]] == VD, Head[Part[z, iiii, 1]] == VU],

(*then*)

TensorStruct = TensorStruct * Part[z, iiii],

(*else*)

If[(*If where is no TS at the beginning then we check TS at the End of the Angle Bracket *)

Or[Head[Part[z, iiii, Length[Part[z, iiii]]]] == VD, Head[Part[z, iiii, Length[Part[z, iiii]]]] == VU],

TensorStruct = TensorStruct * Part[z, iiii, Length[Part[z, iiii]]];

]

]

}](*end of if*)

}](*end of for*);


(*find structure with mask < n_μ * ... > or < ... * n_μ > or < n^μ * ... > or < ... * n^μ > or couple of them*)

For[iiii = 1, iiii<=Length[z], iiii++,

{If[Depth[Part[z, iiii]] > 2, {
```

```
If[(* try to check is where some tensor structure include in AngleBracket *)

Head[Part[z, iiii, 1]] == Times,

(*then*)

{If[(* Try to find tensor structure to the beginning of bracket *)

Or[Head[Part[z, iiii, 1, 1]] ==, Head[Part[z, iiii, 1, 1]] ==, Head[Part[z, iiii, 1, 2]] ==,

Head[Part[z, iiii, 1, 2]] ==],

(*then*)

TensorStruct = TensorStruct * Part[z, iiii];

]

}](*end of if*)

}](*end of first if*)

}](*end of for*);



(* just try to find raw tensor or vector structure *)

For[iiii = 1, iiii<=Length[z], iiii++,

If[(*condition for Raw tensor structures *)

Or[(* metric tensor*)Head[Part[z, iiii]] == MTD, Head[Part[z, iiii]] == MTU,

Head[Part[z, iiii]] == MTM,

(*vector*)Head[Part[z, iiii]] == VD,

Head[Part[z, iiii]] == VU,

(*Tensor*)Head[Part[z, iiii]] == TD,

Head[Part[z, iiii]] == TU, Head[Part[z, iiii]] == TM], (* End of OR Operator *)

TensorStruct = TensorStruct * Part[z, iiii];

]](*end of For*);
```

```
JustSee[TensorStruct]

(*end of For*)]



(* This Procedure make a list of coefficients *)

MakeListOfCoefficients[det1_]:=Module[{iiii, CList, det = DoRaw[det1]},

CList = {};

For[iiii = 1, iiii<=Length[det], iiii++,

If[(*test list for a tensor structures*)MemberQ[CList, IdentifyTensor[Part[det, iiii]]],

(*then*),

(*else*)CList = Join[CList, {IdentifyTensor[Part[det, iiii]]}]

(*end of if*)];

]; (*end of for *)

CList]



(*This Procedure make s Table of Tensor Coefficients...*)

MakeTableOfCoefficients[expression_]:=Module[{zozozo, zuzuzu, iiii},

zozozo = MakeListOfCoefficients[expression];

zuzuzu = DoFull[expression];

For[iiii = 1, iiii<=Length[zozozo], iiii++,

Print[iiii, ") ", zozozo[[iiii]], " ———— ", Coefficient[zuzuzu, zozozo[[iiii]]], " "]

(*end of for*)]](*end of module*)


OutputRule:={(*This rule modify output for reed. Use it only on the final stage.*)

(*VU[k, μ] → k^μ*)VU[k_, μ_]->Superscript[k, μ],
```

```
(*VD[k, μ] → k_μ*)VD[k_, μ_]->Subscript[k, μ],
(*MTD[μ, ν] → g_μν*)MTD[μ_, ν_]->Subscript[g, μν],
(*MTU[μ, ν] → g^μν*)MTU[μ_, ν_]->Superscript[g, μν],
(*MTM[μ, ν] → δ_μ^ν*)MTM[μ_, ν_]->Subsuperscript[δ, μ, ν],
(*TD[n, μ, ν] → n_μν*)TD[n_, μ_, ν_]->Subscript[n, μν],
(*TU[n, μ, ν] → n^μν*)TU[n_, μ_, ν_]->Superscript[n, μν],
(*TM[n, μ, ν] → n_μ^ν*)TM[n_, μ_, ν_]->Subsuperscript[n, μ, ν]
}
```

```
MakeListofEquation[zu_]:=Module[{zuzu, iiii, iii, zo},
EqList = {};
zuzu = DoFull[zu];
zo = MakeListOfCoefficients[zu];
For[iiii = 1, iiii<=Length[zo], iiii++,
If[MemberQ[EqList, Coefficient[zuzu, zo[[iiii]]]], (*then*),
EqList = Join[EqList, {Coefficient[zuzu, zo[[iiii]]]}]
]];
(*Traform into equation list*)
For[iii = 1, iii<=Length[EqList], iii++,
EqList[[iii]] = EqList[[iii]] == 0
];
EqList
]
```

DoFull[expression_]:=

Expand[expression]//.MetricTensorRule//.NScalarProduct//.FullScalarProductRule

//.PartialScalarProductRule//.ComplicateScalarProductRule/.OutputRule

DoRaw[expression_]:=Expand[expression]//.MetricTensorRule//.NScalarProduct

//.FullScalarProductRule//.PartialScalarProductRule//.ComplicateScalarProductRule

JustSee[expression_]:=Expand[expression]/.OutputRule

JustContract[expression_]:=Expand[expression]//.MetricTensorRule//.NScalarProduct

//.OutputRule

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